PhD Tutorial: Unifying the Construction of Various Types of Generalized Coherent States

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Abstract. In this tutorial I intend to present some of the results I obtained through my PhD work in the "quantum optics group of the University of Isfahan" under consideration Dr. R. Roknizadeh and Prof. S. Twareque Ali as my supervisor and advisor, respectively. I will revisit some of the pioneering proposals recently developed the concept of generalized CSs. As it can be observed the customary three generalization methods *(symmetry, algebraic and dynamical)* have never been considered in neither of them. Our intention in this work is at first to investigate the lost ring between the customary three methods and the recently developed ones, as possible. For this purpose it has been devised general analytic descriptions, which successfully demonstrate how different varieties of CSs (which are nonlinear in nature) can be obtained by two processes, first the *"nonlinear CSs"* method and second by *"basis transformations on an underlying Hilbert spaces"*. As a result, I will systematize the recently introduced generalized CSs in a clear and concise way. It will be clear also, that how our results can be considered as a first step in the generation process of the mathematical physics CSs in the context of quantum optics. Besides this, some new results emerge from our studies. I introduce a large classes of generalized CSs, namely the *"dual family"* associated with each set of early known CSs. But, in this relation, the previous processes for constructing the dual pair of Gazeau-Klauder CSs fail to work well, so I outlined a rather different method based on the *"temporal stability"* requirement of generalized nonlinear CSs.

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1. Introduction

Quantum optics is an ideal testing ground for ideas of quantum theory, and coherent states (CSs), first introduced by Shrödinger [92] as venerable objects in physics, are one of the basic axes in researches in this branch of physics. Nowadays much attention is being paid to the CSs and their theoretical generalizations [2, 4, 49, 75], including their experimental generations and applications. The new generalizations have some interesting *"non-classical properties"* such as squeezing, antibunching, sub-Poissonian statistics and oscillatory number distribution. Earlier study of such non-classical effects were regarded as being of interest academically only, but now their applications in quantum communications [90, 91], quantum teleporation [17, 18, 22], dense coding [21], quantum cryptography [18, 47] and detection of gravitational waves [44, 48] are well understood.

It is frequently found in the literature that motivations to generalize the concept of CSs have arisen from *"three methods", symmetry considerations, dynamics and algebraic aspects*. Generalization based on symmetry has led to define CSs for arbitrary Lie groups [23, 24, 25, 78, 79]. Based on dynamics, CSs have been constructed for systems other than harmonic oscillator which has equally spaced energy levels [71, 72, 73, 74] and finally CSs for deformed algebras have been introduced by extending the algebraic definition [10, 61, 89].

But it is also well-known that there are generalizations in very many ways that (in the first sight) have not derived by the above three approaches. To say a few, we can refer to even

and odd CSs [28], Shrödinger cat states [93, 105], all of which have been obtained from some special *superpositions* of canonical CSs or other generalizations. While the canonical CSs do not have any non-classical properties, their superpositions have. Another scheme for generalization of CSs are obtained by using the non-orthogonal bases of a Hilbert space. Part of our works contains the construction of CSs and squeezed state(SSs) using a special set of non-orthogonal but normalizable bases $\{|n\rangle_\lambda, n \in \mathbf{N}\}_{n=0}^{\infty}$ instead of orthonormal one $\{|n\rangle, n \in \mathbf{N}\}_{n=0}^{\infty}$ [83].

In addition to the above generalizations, recently there are some new classes of generalizations of CSs, such as the ones introduced by Klauder-Penson-Sixdeniers (KPS) [52], generalized hypergeometric (HG) CSs [9], Gazeau-Klauder CSs (GK-CSs) [39, 40, 50, 51, 53], Penson-Solomon (PS) CSs [76], including the Mittag-Leffler (ML) CSs [66] and Tricomi (TC) CSs [97] have been constructed by different *"mathematical structures"*. In neither of these works the authors referred to any of the *"three customary methods of generalizations"*. In some of them, even the physical motivations of the three generalizations have been questioned essentially [53].

As is well known, the idea of *"unifying"* the different methods in any field of physics is an old subject which is of interest, as a major challenge for theoretical physicists. The basic aim in this manufacture is to present formalisms to *"unify the different ways of the above scattered models of generalizations of CSs"* [100]. Through doing this, we will arrive at some new families of generalized CSs in the context of quantum optics. In addition to the *elegance* of the present work, this classifications will simplify understanding and the introduction of the theories, more than before. To achieve this purpose, two different approaches will be presented.

(i) *The nonlinear CSs formalism,*
The first approach, basically based on the conjecture that all these states may be studied in the so-called *"nonlinear CSs"* or *"f-deformed CSs"* category [61, 64, 65], the states that attracted much attention in recent years, mostly because they exhibit nonclassical properties. Up to now, many quantum optical states such as q-deformed CSs [61], negative binomial state [101, 102], photon added (and subtracted) CSs [95, 96, 67, 69, 70], the center of mass motion of a trapped ion [64, 65], some nonlinear phenomena such as a hypothetical *"frequency blue shift"* in high intensity photon beams [62] and recently after proposing f-bounded CSs [82], the binomial state (or displaced exited CSs) [83] have been considered as some sorts of nonlinear CSs. Speaking otherwise, the algebraic method may be particularly useful for providing a *unified* treatment of all these states and their *interrelationships*. I attempt now to demonstrate that all sets of KPS and ML CSs, HG, PS and TC CSs, including the two discrete series representations of the group $SU(1,1)$ (both Barut-Girardello (BG) [13] and Gilmore-Perelomov (GP) CSs [45]) and $su(1,1)$-BG CSs for Landau-level (LL) [35] can be classified in the nonlinear CSs with some special nonlinearity function, $f(n)$, by which we may obtain the *"deformed annihilation and creation operators"*, *"generalized displacement operator"* and the *"dynamical Hamiltonian"* corresponding to each class of them. Based on these results, it will be possible to reconstruct all of the above CSs via conventional fashions, i.e. by annihilation and displacement (type) operator definition.

(ii) *The mathematical physics formalism,*
In this approach we attempt to set up a mathematical formalism by which one can produce a vast class of generalized CSs, e.g. binomial states, photon-added CSs and the so-called nonlinear CSs. Therefore, all of the generalized CSs that by the first approach we clarified

the nonlinearity nature's, can also be reconstructed by the second approach [5]. By this formalism we try to establish that a vast class of generalized CSs can be considered as different representations of canonical CSs in the underlying Hilbert spaces. Indeed by defining an appropriate operator $\hat{T}$ (with some special properties) and its action on the canonical CSs one may yield the above generalized CSs. As we will observe later, for the case of nonlinear CSs this operator depends explicitly on the nonlinearity function $f(\hat{n})$. Therefore, in a sense it enables one to obtain some various types of generalized CSs via a *single* mathematical formalism, rather than various distinguishable approaches. Obviously, in each case we have to find an appropriate $\hat{T}$ operator.

In another direction, as a matter of fact introducing the deformed creation and annihilation operators related to the mathematical physics CSs considered in this manufacture may be regarded as a first step in the process of production and detection of these states in the experimental realization schemes of quantum optics. According to the theoretical scheme proposed in Ref. [68], any nonlinear CSs can be generated in a micromaser under intensity-dependent (ID) Jaynes-Cummings model. The authors generalized the "standard" Jaynes-Cummings model to the "multi-photon intensity-dependent" case, which may be defined as a quantum model describing the interaction of a monochromatic electromagnetic field with one two-level atom in a cavity under intensity-dependent coupling through multi-photon transitions. The interaction Hamiltonian of this model can be expressed in the rotating-wave approximation and in the interaction picture as follows ($\hbar = 1$):

$$\hat{H}_{ID}^{(m)} = g\left[a^m f(\hat{n})|a\rangle\langle b| + |b\rangle\langle a| f(\hat{n})(a^\dagger)^m\right], \qquad m = 1, 2, \cdots, \tag{1}$$

where $|a\rangle$ and $|b\rangle$ denote the exited and ground states of atomic level,respectively, g is the coupling constant, a, $a^\dagger$ are the standard photon annihilation and creation operators with algebra $[a, a^\dagger] = \hat{I}$ and $f(\hat{n})$ describes the intensity dependence of atom-field interaction. As is well known $f(\hat{n})$ is the same operator valued (or *"nonlinearity"*) function associated with any class of nonlinear CSs. Although it is pointed in [68] that $f(\hat{n})$ is a real function, in my opinion the formalism can also be extended to any phase-dependent one (the case that I will strict in my researches in the Gazeau-Klauder generalized CSs).

For the special case of one-photon transition the model equation (1) simplify to

$$\hat{H}_{ID}^{(m=1)} = g\left[A|a\rangle\langle b| + |b\rangle\langle a| A^\dagger\right], \qquad m = 1, 2, \cdots, \tag{2}$$

where in the latter equation we have set $A = af(\hat{n})$ and $A^\dagger = a^\dagger f(\hat{n})$, which are the known $f-$deformed ladder operators as the generators of the nonlinear oscillator algebras $\{A, A^\dagger, \hat{H}\}$. *Therefore, as we will observe, the results obtained from the first approach can be considered as an introductory step in the theoretical scheme for "generation of the mathematical physics CSs" have been developed in recent decade.*

In addition to these, some interesting and new remarkable points emerge from the presented results and studies. The first is that both of the two mentioned formalisms provide a framework to construct a vast new families of CSs (named the *dual family*), other than KPS, GH, ML, PS and TC CSs. The second is that the Hamiltonian proposed in [61] and others who cited to him (see e.g. [96]) must be reformed, in view of the *action identity* requirement imposed on the generalized CSs proposed by Gazeau and Klauder [40, 53]. We should quote here that recently some authors (for instance see [30, 31, 32, 33, 34]) have used normal ordering form (factorization) for their Hamiltonians. Indeed they used the *"supersymmetric quantum mechanics"* (SUSQM)

techniques [104] as a *"mathematical tool"* to find the ladder operators for their Hamiltonians. Interestingly our formalism for solvable Hamiltonians gives an easier and clearer manner to obtain these operators whenever necessary.

As we will observe, the above two approaches are not enough to reproduce all the known generalized CSs. GK-CSs is an important case [40]. Applying the first formalism to the GK-CSs, gives readily us a function $f(\hat{n})$ and therefore by the second one we will find an operator $\hat{T}(n)$. But both of these two operators are ill defined, because they depend on a variable γ $(-\infty < \gamma < \infty)$. Other than this defect, we will observe that the dual family of GK-CSs obtained by each of the two approaches are not fully consistent with the Gazeau-Klauder criteria. Strictly speaking it is easily found that they did not satisfy the temporal stability (and therefore the action identity) requirements. To overcome this serious problem, upon using the analytical representations of GK-CSs (denoted by GKCSs), I will introduce an operator $\hat{S}$, depends explicitly on the Hamiltonian of the system, which its action on any nonlinear CSs (which essentially do not possess the temporal stability property), transfers it to a situation that enjoy this property, nicely [85]. In this way, not only we are able to derive the dual family of GKCSs, but also we may obtain the opportunity to introduce the temporal stability version of all the nonlinear CSs, derived in this tutorial and introduced elsewhere.

This tutorial organizes as follows: in section 2 we revisit some of the most important generalized CSs of recent decade, which will be as a necessary tool for our works in later sections. Along unification of mathematical physics (MP) CSs introduced in section 2, section 3 deals with the two unification methods successfully reach this purpose. The generalized displacement operator and the dual family of some of the MP CSs will be presented in section 4. The dual family of GKCSs will be presented in section 5 in a general framework of the temporal stabilization of nonlinear CSs, with some physical appearance of the presented formalism. Finally in section 6 a scheme for the generalized GKCSs and the associated dual family will be offered.

2. Preliminary arguments on mathematical physics (MP) CSs

In this section we will express the explicit form of some generalized CSs which are necessary in our future treatments. Nearly all of these generalized CSs (except the GK-CSs, GKCSs and $su(1,1)$-BG CSs for Landau levels), have been introduced based on some *"mathematical structures"*. So we have called whole of them as *"mathematical physics CSs"*.

2.1. Nonlinear CSs and their dual family

Nonlinear CSs first introduced explicitly in [37, 61, 64, 65], but before them it is implicitly defined by Shanta *etal* [89] in a compact form. This notion attracted much attention in physical literature in recent decade, especially because of their nonclassical properties in quantum optics. Man'ko *etal*'s approach is based on the two following postulates.

The first is that the standard annihilation and creation operators deformed with an intensity dependent function $f(\hat{n})$ (which is an operator valued function), according to the relations:

$$A = af(\hat{n}) = f(\hat{n}+1)a, \qquad A^\dagger = f^\dagger(\hat{n})a^\dagger = a^\dagger f^\dagger(\hat{n}+1), \tag{3}$$

with commutators between A and $A^\dagger$ as

$$[A, A^\dagger] = (\hat{n}+1)f(\hat{n}+1)f^\dagger(\hat{n}+1) - \hat{n}f^\dagger(\hat{n})f(\hat{n}), \tag{4}$$

where a, $a^\dagger$ and $\hat{n} = a^\dagger a$ are bosonic annihilation, creation and number operators, respectively. Ordinarily the phase of f is irrelevant and one may choose f to be real and nonnegative, i.e. $f^\dagger(\hat{n}) = f(\hat{n})$. But to keep general consideration, we take into account the phase dependence of $f(\hat{n})$ in general formalism given here.

The second postulate is that the Hamiltonian of the deformed oscillator in analog to the harmonic oscillator is found to be

$$\hat{H}_{\mathrm{M}} = \frac{1}{2}(AA^\dagger + A^\dagger A), \tag{5}$$

which by Eqs. (3) can be rewritten as

$$\hat{H}_{\mathrm{M}} = \frac{1}{2}\left((\hat{n}+1)f(\hat{n}+1)f^\dagger(\hat{n}+1) + \hat{n}f^\dagger(\hat{n})f(\hat{n})\right), \tag{6}$$

where index M refers to the Hamiltonian as introduced by Man'ko *et al* [61].

The single mode nonlinear CSs obtained as eigen-state of the annihilation operator is as follows:

$$|z, f\rangle = \mathcal{N}_f(|z|^2)^{-1/2} \sum_{n=0}^{\infty} C_n z^n |n\rangle, \tag{7}$$

where the coefficients C_n are given by

$$C_n = \left(\sqrt{[nf^\dagger(n)f(n)]!}\right)^{-1}, \quad C_0 = 1, \quad [f(n)]! \doteq f(n)f(n-1)\cdots f(1), \tag{8}$$

and the normalization constant is determined as $\mathcal{N}_f(|z|^2) = \sum_{n=0}^{\infty} |C_n|^2 |z|^{2n}$. In order to have states belonging to the Fock space, it is required that $0 < \mathcal{N}_f(|z|^2) < \infty$, which implies $|z| \leq \lim_{n\mapsto\infty} n[f(n)]^2$. No further restrictions are then put on $f(n)$. Now with the help of Eqs. (7) and (8) the function $f(n)$ corresponding to any nonlinear CSs is found to be

$$f(n) = \frac{C_{n-1}}{\sqrt{n}C_n}, \tag{9}$$

which plays the key rule in our present work. To recognize the nonlinearity of any CSs we can use this simple and useful relation; by this we mean that if C_n's for any CSs are known, then $f(n)$ can be found from Eq. (9); when $f(n) = 1$ or at most be only a constant phase, we recover the original oscillator algebra, otherwise it is nonlinear.

Recall that by replacing $f(n)$ with $\frac{1}{f(n)}$ in the relations (7) and (8) one immediately gets the nonlinear CSs introduced in [87]. These latter states have been called as the *"dual family"* of nonlinear CSs of Man'ko's type [5].

2.2. Klauder-Penson-Sixdeniers (KPS) and Mittag-Leffler (ML) generalized CSs

Along generalizations of CSs, Klauder, Penson and Sixdeniers introduced the states [52]

$$|z\rangle_{\mathrm{KPS}} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle, \tag{10}$$

where $\rho(n)$ satisfies $\rho(0) = 1$ and the normalization constant is determined as $\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho(n)}$. These states possess three main properties: (i) *normalization* (ii) *continuity in the label* and (iii) *they form an over-complete set* which allows a resolution of the identity with a positive weight function, by appropriately selected functions, $\rho(n)$. The third condition, which is

the most difficult and at the same time the strongest requirements of any sets of CSs, were proved appreciatively by them, through Stieltjes and Hausdorff power moment problem. Explicitly for each particular set of generalized CSs $|z\rangle_{KPS}$, they found the positive weight function $W(|z|^2)$ such that

$$\int\int_{\mathbb{C}} d^2z |z\rangle_{\mathrm{KPS}} W(|z|^2)_{\mathrm{KPS}}\langle z| = \hat{I} = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{11}$$

where $d^2z = |z|d|z|d\theta$. Strictly speaking, evaluating the integral in (11) over θ in the LHS of Eq. (11), setting $|z|^2 \equiv x$ and simplify it, we arrive finally at:

$$\int_0^R x^n W'(x)dx = \rho(n), \qquad n = 0, 1, 2, ..., \qquad 0 < R \leq \infty, \tag{12}$$

where the positive weight functions $W'(x) = \frac{\pi W(x)}{\mathcal{N}(x)}$ must be determined. Then, the authors used the general formalism to a variety of the $\rho(n)$ functions have been defined on the *"whole plane"* and on the *"unit disk"*. In all cases the functions $\rho(n)$ have been choosed such that the resolution of the identity hold.

A special case of these states known as ML CSs may be constructed by replacing the function $\rho(n) = \frac{\Gamma(\alpha n+\beta)}{\Gamma(\beta)}$ in (10), where α , $\beta > 0$ [66].

2.3. Generalized hypergeometric (HG) CSs

A larger class of KPS CSs, can be constructed by starting with the hypergeometric functions [9],

$${}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{x^n}{n!}\ , \tag{13}$$

where α_i and β_i are positive real numbers, q is an arbitrary positive integer and p is restricted by $q-1 \leq p \leq q+1$. (Here $(\gamma)_n$ is the usual Pochhammer symbol, $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1) = \Gamma(\gamma+n)/\Gamma(\gamma)$). The series in (13) converges for all $x \in \mathbb{R}$ if $p = q$ and for all $|x| < 1$ if $p = q+1$. The explicit form of these states are as follows [9]:

$$|z; p, q\rangle_{\mathrm{HG}} = |\alpha_1, \cdots \alpha_p; \beta_1, \cdots \beta_q; z\rangle = {}_p\mathcal{N}_q(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n)}}\ |n\rangle, \tag{14}$$

where ${}_p\rho_q(n)$ are the strictly positive functions of n, defined by the relation

$${}_p\rho_q(n) = {}_p\rho_q(\alpha_1, \cdots \alpha_p; \beta_1, \cdots \beta_q; n) = \Gamma(n+1)\frac{(\beta_1)_n \cdots (\beta_q)_n}{(\alpha_1)_n \cdots (\alpha_p)_n}, \tag{15}$$

and the normalization factor is determined as ${}_p\mathcal{N}_q(|z|^2) = {}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q;\ |z|^2)$.

2.4. Generalized Tricomi (TC) CSs

Tricomi (TC) CSs of the first kind introduced as [97]

$$|z; p\rangle^{(1)}_{\mathrm{TC}} = \mathcal{N}_p(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n! d_p(n)}} |n\rangle, \quad d_p(n) \equiv \frac{p^{-\frac{n}{2}} 2^{-n}}{\sqrt{\pi} e^{\frac{1}{4p}} erfc(\frac{1}{2\sqrt{p}})} \psi\left(\frac{n+1}{2}\ , \frac{1}{2}; \frac{1}{4p}\right), \tag{16}$$

and similarly the second kind of Tricomi's CSs defined as [97]

$$|z;\lambda,\beta\rangle_{\rm TC}^{(2)}=\mathcal{N}_{\lambda,\beta}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!d_{\lambda,\beta}(n)}}|n\rangle,\quad d_{\lambda,\beta}(n)\equiv\frac{\beta^n\psi(n+1,n+2-\lambda;\beta)}{\psi(1,2-\lambda;\beta)},\tag{17}$$

where $p,\beta>0$, λ is arbitrary, $\psi(a,c;z)$ are the Tricomi's confluent hypergeometric functions and the normalization constants $\mathcal{N}_p$ and $\mathcal{N}_{\lambda,\beta}$ may be determined, easily. Both of the latter generalizations introduced by a deviation from the exponential function appears in the weight function (and normalization) of canonical CSs. This idea (which followed in PS CSs will be introduced later) is important, due to the fact that Mandel parameter [58, 59](which demonstrates the statistics of any CSs) depends on the normalization function of any CSs and its derivatives (see for example [52]). So, deviation from exponential function, implies the deviation from Poissonian statistics and forced the CSs to possess the sub(or super) Poissonian statistics.

2.5. Penson-Solomon (PS) generalized CSs:

The generalized CSs introduced by Penson and Solomon (PS) [76] are as follows:

$$|q,z\rangle_{\rm PS}=\mathcal{N}(q,|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{\sqrt{n!}}z^n|n\rangle,\tag{18}$$

where $\mathcal{N}(q,|z|^2)$ is a normalization function, $0\leq q\leq 1$. This definition is based on an entirely analytical prescription, in which the authors proposed the generalized exponential function obtained from the following differential equation:

$$\frac{d\varepsilon(q,z)}{dz}=\varepsilon(q,qz)\Rightarrow\varepsilon(q,z)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{n!}z^n,\tag{19}$$

where $\varepsilon(1,z)=\exp(z)$. This case is also based on the generalization of the exponential function (such as TC CSs).

2.6. Barut-Girardello (BG) and Gilmore-Perelomov (GP) CSs for $su(1,1)$ Lie algebra:

As an important generalized CSs one may refer to the BG CSs, defined for the discrete series representations of the group $SU(1,1)$ [13]. These states can be realized in some physical systems such as the Pöschl-Teller and infinite square well potentials. The BG CSs decomposed over the number-state bases as:

$$|z,\kappa\rangle_{\rm BG}=\mathcal{N}_{\rm BG}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!\Gamma(n+2\kappa)}}|n\rangle,\quad z\in\mathbb{C}\ ,\tag{20}$$

where $\mathcal{N}_{\rm BG}$ is a normalization constant and the label κ takes the values $1,3/2,2,5/2,...$ labels the $SU(1,1)$ representation being used.

Similarly the GP CSs defined as:

$$|z,\kappa\rangle_{\rm GP}=\mathcal{N}_{\rm GP}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\sqrt{\frac{\Gamma(n+2\kappa)}{n!}}z^n|n\rangle,\qquad |z|<1,\tag{21}$$

where again $\mathcal{N}_{\rm GP}$ is a normalization factor and κ takes the same values, as in BG CSs.

2.7. su(1,1)-Barut-Girardello (BG) CSs for Landau levels:

As an example more close to physics, it is well-known that the Landau levels is directly related to quantum mechanical study of the motion of a charged and spinless particle on a flat plane in a constant magnetic field [41, 55] . Recently it is realized two distinct symmetries corresponding to these states, namely $su(2)$ and $su(1,1)$ in [35]. The author showed that the quantum states of the Landau problem corresponding to the motion of a spinless charged particle on a flat surface in a constant magnetic field $\beta/2$ along $z-$axis may be obtained as:

$$|n,m\rangle = \frac{e^{im\varphi}}{\sqrt{2\pi}}(\frac{r}{2})^{\frac{2\alpha+1}{2}} e^{-\beta r^2/8} L_{n,m}^{(\alpha,\beta)}(\frac{r^2}{4}), \tag{22}$$

where $0 \leq \varphi \leq 2\pi$, $\alpha > -1, n \geq 0, 0 \leq m \leq n$ and $L_{n,m}^{(\alpha,\beta)}$ are the associated Laguerre functions. Constructing the Hilbert space spanned by $\mathfrak{H} := \{|n,m\rangle\}_{n\geq 0, 0\leq m \leq n}$, there it is shown that the BG CSs associated to this system can be obtained as the following combination of the orthonormal basis:

$$|z\rangle_m = \frac{|z|^{(\alpha+m)/2}}{\sqrt{I_{\alpha+m}(2|z|)}} \sum_{n=m}^{\infty} \frac{z^{n-m}}{\sqrt{\Gamma(n-m+1)\Gamma(\alpha+n+1)}}|n,m\rangle, \tag{23}$$

where $I_{\alpha+m}(2|z|)$ is the modified Bessel function of the first kind [103]. The states in (23) derived with the help of the lowering generator of the $su(1,1)$ Lie algebra, the action of which defined by the relation:

$$K_-|n,m\rangle = \sqrt{(n+\alpha)(n-m)}|n-1,m\rangle, \qquad K_-|m,m\rangle = 0. \tag{24}$$

2.8. Gazeau-Klauder generalized CSs (GK-CSs)

Adopting certain physical criteria rather than imposing selected mathematical requirements, Klauder and Gazeau by reparametrizing the generalized CSs $|z\rangle$ in terms of a two independent parameters J and γ, introduced the generalized CSs $|J,\gamma\rangle$, known ordinarily as Gazeau-Klauder CSs (we denoted them by GK-CSs) in the physical literature [40, 53]. These states are explicitly defined by the expansion

$$|J,\gamma\rangle = \mathcal{N}(J)^{-1/2} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-ie_n\gamma}}{\sqrt{\rho(n)}}|n\rangle, \tag{25}$$

where the normalization constant is given by $\mathcal{N}(J) = \sum_{n=0}^{\infty} \frac{J^n}{\rho(n)}$, and $\rho(n)$ is a positive weight factor with $\rho(0) \equiv 1$ by convention and the domains of J and γ are such that $J \geq 0$ and $-\infty < \gamma < \infty$. These states required to satisfy the following properties: (i) *continuity of labeling*: if $(J,\gamma) \to (J',\gamma')$ then, $\||J,\gamma\rangle - |J',\gamma'\rangle\| \to 0$, (ii) *resolution of the identity*: $\hat{I} = \int |J,\gamma\rangle\langle J,\gamma| d\mu(J,\gamma)$ as usual and two extra properties: (iii) *temporal stability*: $\exp(-i\hat{H}t)|J,\gamma\rangle = |J,\gamma+\omega t\rangle$ and (iv) *the action identity*: $H = \langle J,\gamma|\hat{H}|J,\gamma\rangle = \omega J$, where H and $\hat{H}$ are classical and quantum mechanical Hamiltonians of the system, respectively. It must be understood that the forth condition forced the generalized CSs to have the essential property: *"the most classical quantum states"*, but now in the sense of *"energy"* of the dynamical system, in the same way that the canonical CSs is a quantum state which its position and momentum expectation values obey the classical orbits of harmonic oscillator in phase space. It must also

be noted that for the second criteria, Gazeau and Klauder defined

$$\int |J,\gamma\rangle\langle J,\gamma| d\mu(J,\gamma) = \lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma} d\gamma \int_0^R dJ\mathcal{N}(J)\varrho(J)|J,\gamma\rangle\langle J,\gamma|$$
$$= \sum_{n=0}^{\infty}|n\rangle\langle n|\int_0^R dJ\,\varrho(J)\frac{J^n}{\rho(n)} = \sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{I}\ . \qquad (26)$$

This condition finally resulted in the moment problem as follows:

$$\int_0^R dJ J^n \varrho(J) = \rho(n)\cdot \qquad (27)$$

In Eq. (25) the kets $|n\rangle$ are the eigen-vectors of the Hamiltonian $\hat{H}$, with the eigen-energies E_n,

$$\hat{H}|n\rangle = E_n|n\rangle \equiv \hbar\omega e_n|n\rangle \equiv e_n|n\rangle, \quad \hbar\equiv 1, \quad \omega\equiv 1, \quad n=0,1,2,...\ , \qquad (28)$$

where the re-scaled spectrum e_n, satisfied the inequalities

$$0 = e_0 < e_1 < e_2 < \cdots < e_n < e_{n+1} < \cdots\ . \qquad (29)$$

The action identity *uniquely* specified $\rho(n)$ in terms of the eigen-values of the Hamiltonian $\hat{H}$

$$\rho(n) = \Pi_{k=1}^{n} e_k \equiv [e_n]!\ . \qquad (30)$$

As an example, for the shifted Hamiltonian of harmonic oscillator we have the canonical CSs in the language of Gazeau-Klauder, denoted by $|J,\gamma\rangle_{CCS}$:

$$|J,\gamma\rangle_{CCS} = e^{-J/2}\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-in\gamma}}{\sqrt{n!}}|n\rangle. \qquad (31)$$

Eq. (30) obviously states that the functions $\rho(n)$ is directly related to the spectrum of the dynamical system. So every Hamiltonian *uniquely* determined the associated CS, although the inverse is not true [36, 83].

2.8.1. Analytical representations of Gazeau-Klauder generalized CSs (GKCSs) El Kinani and Daoud, in a series of papers [30, 31, 32, 33, 34] imposed a minor modification on the GK-CSs in (25) via generalizing the Bargman representation for the standard harmonic oscillator [12]. Following the path of Gazeau-Klauder, the authors introduced the *analytical representations of GK-CSs*, denoted by us as GKCSs:

$$|z,\alpha\rangle \doteq \mathcal{N}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n e^{-i\alpha e_n}}{\sqrt{\rho(n)}}|n\rangle, \qquad z\in\mathbb{C}, \qquad \alpha\in\mathbb{R}, \qquad (32)$$

where the normalization constant is given by $\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{\rho(n)}$ and the function $\rho(n)$ can be expressed in terms of the eigen-values as $\rho(n) = [e_n]!$. Briefly speaking, they replaced $-\infty < \gamma < \infty$ and $J > 0$ in (25) by $\alpha\in\mathbb{R}$ and $z\in\mathbb{C}$, respectively. We must emphasize the main difference between the GKCSs presented in (32) and GK-CSs in (25) in view of the significance and the role of γ and α, particularly in the integration procedure, in order to establish the resolution of the unity. According to their proposal for this purpose it is required to find an appropriate positive measure $d\lambda(z)$ such that the following integral satisfied:

$$\int_0^R |z,\alpha\rangle\langle z,\alpha| d\lambda(z) = \sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{I}, \qquad 0 < R \leq \infty. \qquad (33)$$

Inserting (32) in (33), writing $z = xe^{i\theta}$ and then expressing the measure as:

$$d\lambda(z) = d\lambda(|z|^2) = \pi\mathcal{N}(x^2)\sigma(x^2)xdxd\theta, \tag{34}$$

performing the integration over $\theta \in [0, 2\pi]$, the over-completeness relation (33) finally boils down to the moment problem (see [52] and Refs. therein, especially [1]):

$$\int_0^R x^n\sigma(x)dx = \rho(n). \tag{35}$$

Note that while we have integrated on γ in (26), this is not hold for α parameter in the procedure led to (35).

3. The unification methods

In what follows two approaches will be presented which by them I try to come back to the one (or more) of the three customary generalization methods of constructing the MP CSs we revisited in section 2, as well as some others. As mentioned in the introduction, while these *"unification methods"* have been established, *"some new classes of generalized CSs"* will be obtained. Indeed, we duplicate the number of them through introducing the dual family associated with each classes of them.

3.1. The first approach: the nonlinear CSs method

This approach is basically based on the conjecture that all classes of the generalized CSs outlined in section 2 can be interpreted as nonlinear CSs.

Example 1 *KPS and ML CSs as nonlinear CSs*

To start with, we demonstrate the relation between the KPS (and therefore ML CSs) and nonlinear CSs. Looking at the KPS CSs in (10) and using (9) yields the nonlinearity function as:

$$f_{\text{KPS}}(\hat{n}) = \sqrt{\frac{\rho(\hat{n})}{\hat{n}\rho(\hat{n}-1)}}, \tag{36}$$

which provides simply a bridge between KPS and nonlinear CSs. As a special case when $\rho(n) = n!$, i.e. the canonical CSs, we obtain $f_{\text{KPS}}(n) = 1$.

From Eq. (30) e_n can easily be found in terms of $\rho(n)$:

$$e_n = \frac{\rho(n)}{\rho(n-1)}. \tag{37}$$

Remembering that e_ns are the eigen-values of the Hamiltonian, it will be obvious that neither every $\rho(n)$ of KPS CSs nor every $f_{\text{KPS}}(n)$ of nonlinear CSs are physically acceptable, when the dynamics of the system (Hamitonian) is specified. Using Eqs. (36) and (37) we get

$$f_{\text{KPS}}(n) = \sqrt{\frac{e_n}{n}} \Leftrightarrow e_n = n(f_{\text{KPS}}(n))^2. \tag{38}$$

Now by using (36) we are able to find $f(\hat{n})$ for all sets of KPS CSs (on the whole plane and on a unit disk) and then the deformed annihilation and creation operators $A = af_{KPS}(\hat{n})$ and

$A^\dagger = f^\dagger_{KPS}(\hat{n})a^\dagger$ may easily be obtained. Returning to the above descriptions, the operators A and $A^\dagger$ satisfy the following relations :

$$A|n\rangle = \sqrt{e_n}|n-1\rangle, \tag{39}$$

$$A^\dagger|n\rangle = \sqrt{e_{n+1}}|n+1\rangle, \tag{40}$$

$$[A, A^\dagger]|n\rangle = (e_{n+1} - e_n)|n\rangle, \quad [A, \hat{n}] = A, \quad [A^\dagger, \hat{n}] = -A^\dagger. \tag{41}$$

Also note that $A^\dagger A|n\rangle = e_n|n\rangle$, not equal to $n|n\rangle$ in general. With these results in mind, it would be obvious that Eq. (38) is not consistent with the relations (5) and (6) for the Hamiltonian. A closer look at Eq. (38) which is a consequence of imposing the action identity on the Hamiltonian of the system leads us to obtain a new form of the Hamiltonian for the nonlinear CSs as

$$\hat{H}|n\rangle = nf^2(n)|n\rangle \Leftrightarrow \hat{H} = \hat{n}f^2(\hat{n}) = A^\dagger A. \tag{42}$$

Consequently, this form of the Hamiltonian may be considered as *"normal-ordered"* of the Man'ko *et al* Hamiltonian H_{M}, introduced in Eq. (5), $\hat{H} = \,:\hat{H}_M:$. Therefore the associated Hamiltonian for the KPS CSs can be written as

$$\hat{H}_{\mathrm{KPS}} = \hat{n}(f_{\mathrm{KPS}}(\hat{n}))^2 = \frac{\rho(\hat{n})}{\rho(\hat{n}-1)}. \tag{43}$$

Comparing Eqs. (5) and (6) with the Eqs. (38), (42), implies that if we require that the KPS and nonlinear CSs possess the action identity property, the associated Hamiltonian when expressed in terms of ladder operators must be reformed in normal-ordered form.

In summary our considerations enable one to obtain f-deformed annihilation and creation operators (and also as we will see in later sections, the generalized displacement operators) as well as the Hamiltonian for all sets of $|z\rangle_{\mathrm{KPS}}$ discussed in Ref. [52] (on the whole plane and on a unit disk), after demonstrating that for each of them there exist a special nonlinearity function $f(\hat{n})$ (for the detail of results see [84]).

Example 2 *HG CSs as nonlinear CSs*

Applying the presented formalism in this section on the HG CSs in (14) yields the following nonlinearity function

$$_pf_q(\hat{n}) = {_pf_q}(\alpha_1, \cdots \alpha_p; \beta_1, \cdots \beta_q; \hat{n}) = \sqrt{(\hat{n}-1)\frac{(\beta_1+\hat{n}-1)\cdots(\beta_q+\hat{n}-1)}{(\alpha_1+\hat{n}-1)\cdots(\alpha_p+\hat{n}-1)}}\,, \tag{44}$$

and the dynamical Hamiltonian corresponding to such systems as

$$\hat{H}(\hat{n}) = \hat{n}\,{_pf_q^2}(\hat{n}) = \hat{n}(\hat{n}-1)\frac{(\beta_1+\hat{n}-1)\cdots(\beta_q+\hat{n}-1)}{(\alpha_1+\hat{n}-1)\cdots(\alpha_p+\hat{n}-1)}\,, \tag{45}$$

which is clearly $\hat{n}$-dependent.

Example 3 *TC CSs as nonlinear CSs*

The Tricomi (TC) CSs of the first kind introduced in (16). For the nonlinearity functions responsible to these states, as well as the dynamical Hamiltonians of such system we get

$$f^{(1)}_{\mathrm{TC}}(\hat{n}) = \sqrt{\frac{2}{\sqrt{p}}\,\frac{\psi(\frac{\hat{n}+1}{2};\frac{1}{2};\frac{1}{4p})}{\psi(\frac{\hat{n}}{2};\frac{1}{2};\frac{1}{4p})}}, \qquad \hat{H}^{(1)}(\hat{n}) = \hat{n}\frac{2}{\sqrt{p}}\,\frac{\psi(\frac{\hat{n}+1}{2};\frac{1}{2};\frac{1}{4p})}{\psi(\frac{\hat{n}}{2};\frac{1}{2};\frac{1}{4p})}\,. \tag{46}$$

Similarely for the second kind of Tricomi's CSs defined in (17) we find

$$f^{(2)}_{\mathrm{TC}}(\hat{n}) = \sqrt{\beta\, \frac{\psi(\hat{n}+1;\hat{n}+2-\lambda;\beta)}{\psi(\hat{n};\hat{n}+1-\lambda;\beta)}}, \qquad \hat{H}^{(2)}(\hat{n}) = \hat{n}\beta\, \frac{\psi(\hat{n}+1;\hat{n}+2-\lambda;\beta)}{\psi(\hat{n};\hat{n}+1-\lambda;\beta)}\,. \tag{47}$$

as the nonlinearity function and the Hamiltonian which are clearly $\hat{n}$-dependent. Therefore, again setting $f^{(1)}_{\mathrm{TC}}$ and $f^{(2)}_{\mathrm{TC}}$ in (3) one can easily gets the ladder operators responsible to such systems of CSs and both of them can be obtained by annihilation operator definition.

Example 4 *PS generalized CSs as nonlinear CSs*

As another example we can simply deduce the nonlinearity function for the generalized CSs introduced by Penson and Solomon (PS) [76]. As it may be observed, in the construction of PS CSs (see Eq. (18)), the authors have not used the ladder(or displacement) operator definition for their states. But the present formalism helps one to find the nonlinearity function (and so the annihilation and creation operators evolve in these states) as

$$f_{\mathrm{PS}}(\hat{n}) = q^{1-\hat{n}}. \tag{48}$$

Therefore the method enables one to reproduce them through solving the eigen-value equation $A|z,q\rangle_{PS} = aq^{1-\hat{n}}|z,q\rangle_{PS} = z|z,q\rangle_{PS}$, in addition to introducing a $\hat{n}$-dependent Hamiltonian describing the dynamics of the system:

$$\hat{H}_{\mathrm{PS}}(\hat{n}) = \hat{n}q^{2(1-\hat{n})}. \tag{49}$$

Example 5 *BG CSs for $su(1,1)$ Lie algebra as nonlinear CSs*

Using (9) for BG CSs in (20), one may find

$$f_{\mathrm{BG}}(\hat{n}) = \sqrt{\hat{n}+2\kappa-1}, \quad n = 0,1,2,\dots\,. \tag{50}$$

So, an $\hat{n}$-dependent Hamiltonian describing the dynamics of the system immediately obtained as:

$$\hat{H}_{\mathrm{BG}}(\hat{n}) = \hat{n}(\hat{n}+2\kappa-1), \tag{51}$$

with $\hat{H}_{\mathrm{BG}}|n\rangle = e_n|n\rangle$; When $\kappa = 3/2$ and $\kappa = \lambda+\eta$ (λ and η are two parameter characterize the Pösch-Teller potential with $\kappa = [(\lambda+\eta+1)/2] > 3/2$) we obtain the infinite square well and Pöschl-Teller potentials, respectively [7]. Therefore the dynamical group associated with these two potentials is the $SU(1,1)$ group have been established. Also, if one takes into account the action of $A = af_{\mathrm{BG}}(\hat{n})$, $A^\dagger = f_{\mathrm{BG}}(\hat{n})a^\dagger$ and $[A, A^\dagger]$ on the states $|\kappa,n\rangle$ we obtain

$$A|\kappa,n\rangle = \sqrt{n(n+2\kappa-1)}|\kappa,n-1\rangle, \tag{52}$$

$$A^\dagger|\kappa,n\rangle = \sqrt{(n+2\kappa)(n+1)}|\kappa,n+1\rangle, \tag{53}$$

$$[A, A^\dagger]|\kappa,n\rangle = (n+\kappa)|\kappa,n\rangle. \tag{54}$$

We may conclude that the generators of $su(1,1)$ algebra $\hat{L}_-, \hat{L}_+, \hat{L}_{12}$ can be expressed in terms of the deformed annihilation and creation operators including their commutators such that

$$\begin{aligned} \hat{L}_- &\equiv \frac{1}{\sqrt{2}}A = \frac{1}{\sqrt{2}}af_{\mathrm{BG}}(\hat{n}), \\ \hat{L}_+ &\equiv \frac{1}{\sqrt{2}}A^\dagger = \frac{1}{\sqrt{2}}f_{\mathrm{BG}}(\hat{n})a^\dagger, \\ \hat{L}_{12} &\equiv \frac{1}{2}[A, A^\dagger]. \end{aligned} \tag{55}$$

Note that the presented approach not only recovers the results of Ref. [7] in a simpler and clearer manner, but also it gives the explicit form of the operators $\hat{L}_-, \hat{L}_+, \hat{L}_{12}$ as some *"intensity dependent"* operators in consistence with the Holstein-Primakoff single mode realization of the $su(1,1)$ Lie algebra [43]. Obviously, similar discussion can be followed for the GP representations of $SU(1,1)$ group.

Example 6 *$su(1,1)$-BG CSs for Landau levels as nonlinear CSs*

BG CSs associated with algebra $su(1,1)$ for Landau levels have been derived in Ref. (23). A deeply inspection to these states, in comparison to the states in (20) shows a little difference, in view of the lower limit of the summation sign in the former Eq. (23). But this situation is similar to the states known as photon-added CSs [6] in the sense that both of them are combinations of Fock space, with a cut-off in the summation from below. This common feature leads one to go on with the same procedure that has been done already to yield the nonlinearity function of the photon-added CSs in [95]. Let define the deformed annihilation operator and the nonlinearity function similar to photon-added CSs as

$$A = f(\hat{n})a, \qquad f(n) = \frac{C_n}{\sqrt{n+1}C_{n+1}}, \tag{56}$$

where the states (7) have been used (replacing $|n\rangle$ by $|n,m\rangle$) as the eigen-states of the new annihilation operator defined in Eq. (56). Upon these considerations one can calculate the nonlinearity function for the $su(1,1)$-BG CSs related to Landau level as follows:

$$f_{\mathrm{LL}}(\hat{n}) = \frac{(\hat{n}-m+1)(\hat{n}+\alpha+1)}{\sqrt{\hat{n}+1}}. \tag{57}$$

Hence again the ladder operators corresponding to this system may be easily obtained with the usage of (3). Similar to previous cases, the $\hat{n}$-dependent Hamiltonian describing the dynamics of the system obtained as:

$$\hat{H}(\hat{n}) = \hat{n}\,\frac{(\hat{n}-m+1)^2(\hat{n}+\alpha+1)^2}{\hat{n}+1}. \tag{58}$$

Two points must be emphasized here, at the end of this subsection. The first is that this first approach (as well as the second one will be explained later) does not works well for GK-CSs and GKCSs, so attention will be paied to these cases in future sections 5 and 6. The second is that, although the deformed annihilation and creation operators have not been introduced explicitly in neither of the foregoing examples, finding these operators when the nonlinearity functions have been introduced is very clear using Eq. (3).

Therefore, to this end all of the discussed mathematical physics CSs may be reconstructed by *"annihilation operator definition"*, i.e. $A_f|z,f\rangle_{\mathrm{MP}} = z|z,f\rangle_{\mathrm{MP}}$, where A_f and $|z,f\rangle_{\mathrm{MP}}$ are the $f-$deformed annihilation operators associated with each class of MP CSs, and the MP CSs introduced in this subsection, respectively. So, the algebraic definition of MP CSs has been established, clearly. Besides, we arrive at an $n-$dependent Hamiltonian related to each set of MP CSs, which describes the dynamics of the corresponding system. The third definition, the group theoretical definition and the generalized displacement operator, will be discussed in section 4. Altogether in this manner *we obtain the lost ring between the MP CSs and the three generalization methods, successfully.*

3.2. The second approach: the general setting of mathematical physics method

The primary object for this discussion will be an abstract Hilbert space $\mathfrak{H}$. Let T be an operator on this space with the properties [5]

(i) $\hat{T}$ is densely defined and closed; we denote its domain by $\mathcal{D}(T)$.

(ii) $\hat{T}^{-1}$ exists and is densely defined, with domain $\mathcal{D}(T^{-1})$.

(iii) The vectors $\phi_n \in \mathcal{D}(\hat{T}) \cap \mathcal{D}(\hat{T}^{-1})$ for all n and there exist non-empty open sets $\mathcal{D}_{\hat{T}}$ and $\mathcal{D}_{\hat{T}^{-1}}$ in $\mathbb{C}$ such that $\eta_z \in \mathcal{D}(\hat{T}), \forall z \in \mathcal{D}_{\hat{T}}$ and $\eta_z \in \mathcal{D}(\hat{T}^{-1}), \forall z \in \mathcal{D}_{\hat{T}^{-1}}$.

Note that condition (1) implies that the operator $\hat{T}^*\hat{T} = \hat{F}$ is self adjoint.

Let

$$\phi_n^F := \hat{T}^{-1}\phi_n \, , \qquad \phi_n^{F^{-1}} := \hat{T}\phi_n \, , \qquad n = 0, 1, 2, \ldots, \infty \, ; \tag{59}$$

we define the two new Hilbert spaces:

(i) $\mathfrak{H}_F$, which is the completion of the set $\mathcal{D}(\hat{T})$ in the scalar product

$$\langle f|g\rangle_F = \langle f|\hat{T}^*\hat{T}g\rangle_{\mathfrak{H}} = \langle f|\hat{F}g\rangle_{\mathfrak{H}}. \tag{60}$$

The set $\{\phi_n^F\}$ is orthonormal in $\mathfrak{H}_F$ and the map $\phi \longmapsto \hat{T}^{-1}\phi, \ \phi \in \mathcal{D}(\hat{T}^{-1})$ extends to a *unitary* map between $\mathfrak{H}$ and $\mathfrak{H}_F$. If both $\hat{T}$ and $\hat{T}^{-1}$ are bounded, $\mathfrak{H}_F$ coincides with $\mathfrak{H}$ as a set. If $\hat{T}^{-1}$ is bounded, but $\hat{T}$ is unbounded, so that the spectrum of $\hat{F}$ is bounded away from zero, then $\mathfrak{H}_F$ coincides with $\mathcal{D}(\hat{T})$ as a set.

(ii) $\mathfrak{H}_{F^{-1}}$, which is the completion of $\mathcal{D}(\hat{T}^{*-1})$ in the scalar product

$$\langle f|g\rangle_{F^{-1}} = \langle f|\hat{T}^{-1}\hat{T}^{*-1}g\rangle_{\mathfrak{H}} = \langle f|\hat{F}^{-1}g\rangle_{\mathfrak{H}}. \tag{61}$$

The set $\{\phi_n^{F^{-1}}\}$ is orthonormal in $\mathfrak{H}_{F^{-1}}$ and the map $\phi \longmapsto \hat{T}\phi, \ \phi \in \mathcal{D}(\hat{T})$ extends to a *unitary* map between $\mathfrak{H}$ and $\mathfrak{H}_{F^{-1}}$. If the spectrum of $\hat{F}$ is bounded away from zero, then $\hat{F}^{-1}$ is bounded and one has the inclusions

$$\mathfrak{H}_F \subset \mathfrak{H} \subset \mathfrak{H}_{F^{-1}} \, . \tag{62}$$

The spaces $\mathfrak{H}_F$ and $\mathfrak{H}_{F^{-1}}$ will refer to as a *dual pair* and when (62) is satisfied, the three spaces $\mathfrak{H}_F$, $\mathfrak{H}$ and $\mathfrak{H}_{F^{-1}}$ will be called a *Gelfand triple* [42]. (Actually, this is a rather simple example of a Gelfand triple, consisting only of a triplet of Hilbert spaces [8]).

On $\mathfrak{H}$ we take the operators $a, a^\dagger, \hat{n} = a^\dagger a$:

$$a\phi_n = \sqrt{n}\phi_{n-1}, \quad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}, \quad \hat{n}\phi_n = n\phi_n \, . \tag{63}$$

These operators satisfy:

$$[a, a^\dagger] = \hat{I}, \quad [a, \hat{n}] = a, \quad [a^\dagger, \hat{n}] = -a^\dagger \, . \tag{64}$$

On $\mathfrak{H}_F$ we have the transformed operators:

$$a_F = \hat{T}^{-1}a\hat{T}, \quad a_F^\dagger = \hat{T}^{-1}a^\dagger\hat{T}, \quad \hat{n}_F = T^{-1}\hat{n}\hat{T} \, . \tag{65}$$

These operators satisfy the same commutation relations as $a, a^\dagger$ and $\hat{n}$:

$$[a_F, a_F^\dagger] = \hat{I}, \quad [a_F, \hat{n}_F] = a_F, \quad [a_F^\dagger, \hat{n}_F] = -a_F^\dagger \ . \tag{66}$$

Also on $\mathfrak{H}_F$

$$a_F\phi_n^F = \sqrt{n}\phi_{n-1}, \quad a_F^\dagger\phi_n^F = \sqrt{n+1}\phi_{n+1}^F, \quad \hat{n}_F\phi_n^F = n\phi_n^F \ . \tag{67}$$

Clearly, considered as operators on $\mathfrak{H}_F$, a_F and $a_F^\dagger$ are adjoints of each other and indeed they are just the unitary transforms on $\mathfrak{H}_F$ of the operators a and $a^\dagger$ on $\mathfrak{H}$. On the other hand, if we take the operator a_F, let it act on $\mathfrak{H}$ and look for its adjoint on $\mathfrak{H}$ under this action, we obtain by (65) the operator $a^\sharp = \hat{T}^* a^\dagger \hat{T}^{*-1}$ which, in general, is different from $a_F^\dagger$ and also $[a_F, a^\sharp] \neq \hat{I}$, in general. In an analogous manner, we shall define the corresponding operators $a_{F^{-1}}, a_{F^{-1}}^\dagger, \hat{n}_F$, on $\mathfrak{H}_{F^{-1}}$ with their related actions on $\{\phi_n^{F^{-1}}\}_{n=0}^\infty$.

Therefore, three unitarily equivalent sets of operators have been obtained: $a, a^\dagger, \hat{n}$, defined on $\mathfrak{H}$, $a_F, a_F^\dagger, \hat{n}_F$, defined on $\mathfrak{H}_F$ and $a_{F^{-1}}, a_{F^{-1}}^\dagger, \hat{n}_{F^{-1}}$ defined on $\mathfrak{H}_{F^{-1}}$. On their respective Hilbert spaces, they define under commutation the standard oscillator Lie algebra. On the other hand, if they are all considered as operators on $\mathfrak{H}$, the algebra generated by them and their adjoints on $\mathfrak{H}$ (under commutation) is, in general, very different from the oscillator algebra and could even be an infinite dimensional Lie algebra.

Writing $A = a_F$, $A^\dagger = a^\sharp$, both considered as operators on $\mathfrak{H}$, if they satisfy the relation

$$AA^\dagger - \lambda A^\dagger A = C(\hat{n}), \tag{68}$$

where $\lambda \in \mathbb{R}_*^+$ is a constant and $C(\hat{n})$ is a function of the operator $\hat{n}$, then the three operators A, $A^\dagger$, $\hat{H} = (1/2)(AA^\dagger + A^\dagger A)$ or according to (42) its normal ordered form, are said to generate a *"generalized oscillator algebra"* or *"deformed oscillator algebra"* [20]. Note that on $\mathfrak{H}$, A and $A^\dagger$ are adjoints of each other.

3.3. Construction of generalized CSs

Considering a somewhat more general notation and writing the canonical CSs as,

$$|z\rangle = \eta_z = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, \phi_n, \quad \forall z \in \mathbb{C} \ , \tag{69}$$

defined as vectors in an abstract (complex, separable) Hilbert space $\mathfrak{H}$, for which the vectors ϕ_n form an orthonormal basis $\langle \phi_n | \phi_m \rangle_{\mathfrak{H}} = \delta_{nm}, \ \ n, m = 0, 1, 2, \cdots, \infty$. Then consider the vectors

$$\eta_z^F = \hat{T}^{-1}\eta_z = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, \phi_n^F \tag{70}$$

on $\mathfrak{H}_F$. These are the images of the η_z in $\mathfrak{H}_F$ and are the normalized canonical CSs on this Hilbert space (recall that the vectors ϕ_n^F are orthonormal in $\mathfrak{H}_F$). Similarly, define the vectors

$$\eta_z^{F^{-1}} = \hat{T}\eta_z = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, \phi_n^{F^{-1}} \ , \tag{71}$$

as the canonical CSs η_z unitarily transported from $\mathfrak{H}$ to $\mathfrak{H}_{F^{-1}}$.

Let me now consider the η_z^F as being vectors in $\mathfrak{H}$ and similarly the vectors $\eta_z^{F^{-1}}$ also as vectors in $\mathfrak{H}$. To what extent can we then call them (generalized) CSs? Specifically, we would like to find an orthonormal basis $\{\psi_n\}_{n=0}^\infty$ in $\mathfrak{H}$ and a transformation $w = f(z)$ of the complex plane to itself such that:

(a) one could write,

$$\eta_z^F = \zeta_w = \mathcal{N}'(|w|^2)^{-1/2}\, \Omega(w) \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{[x_n!]}}\, \psi_n \ , \tag{72}$$

where $\mathcal{N}'$ is a new normalization constant, $\Omega(w)$ is a phase factor and $\{x_n\}_{n=1}^{\infty}$ is a sequence of non-zero positive numbers, to be determined;

(b) there should exist a measure $d\lambda(\rho)$ on $\mathbb{R}^+$, such that with respect to the measure $d\mu(w,\overline{w}) = d\lambda(\rho)\, d\vartheta$ (where $w = \rho e^{i\vartheta}$) the resolution of the identity,

$$\int_{\mathcal{D}} |\zeta_w\rangle\langle\zeta_w| \mathcal{N}'(|w|^2) d\mu(w,\overline{w}) = \hat{I} \ , \tag{73}$$

would hold on $\mathfrak{H}$ (as is the case with the canonical coherent states). Here again, $\mathcal{D}$ is the domain of the complex plane, $\mathcal{D} = \{w \in \mathbb{C} \mid |w| < L\}$, where $L^2 = \lim_{n\to\infty} x_n$.

A general answer to the above question may be hard to find. But several classes of examples will be presented below, all physically motivated, for which the above construction can be carried out. These include in particular all the so called nonlinear as we will observe in this tutorial and the deformed and squeezed CSs (see for detail our work [5]), which appear so abundantly in the quantum optical and physical literature [61, 75, 94].

Whenever the two sets of vectors $\{\eta_z^F\}$ and $\{\eta_z^{F^{-1}}\}$ form CSs families in the above sense, we shall call them a *dual pair*.

3.4. Examples of the general construction of CSs

3.4.1. Photon-added and binomial states as bases Let $\hat{T}$ be an operator such that $\hat{T}^{-1}$ has the form

$$\hat{T}^{-1} = e^{\lambda a^\dagger} G(a), \tag{74}$$

where $\lambda \in \mathbb{R}$ and $G(a)$ is a function of the operator a such that $\hat{T}$ and $\hat{T}^{-1}$ satisfy the postulated conditions (1)-(3) of Section 3.2. We can compute the two transformed operators a_F and $a_F^\dagger$ on $\mathfrak{H}_F$ ($\hat{F} = \hat{T}^*\hat{T} = e^{-\lambda a} G(a^\dagger)^{-1} G(a)^{-1} e^{-\lambda a^\dagger}$) to be:

$$a_F = \hat{T}^{-1} a \hat{T} = a - \lambda\hat{I} \ , \qquad a_F^\dagger = \hat{T}^{-1} a^\dagger \hat{T} = G(a - \lambda\hat{I}) a^\dagger G(a - \lambda\hat{I})^{-1} \ . \tag{75}$$

Thus, since a commutes with $G(a - \lambda\hat{I})$, we obtain $[a_F, a_F^\dagger] = \hat{I}$, as expected. The two operators $A = a_F$ and $A^\dagger = \hat{T}^* a^\dagger \hat{T}^{*^{-1}}$, defined on $\mathfrak{H}$, are

$$A = a - \lambda\hat{I}, \quad A^\dagger = a^\dagger - \lambda\hat{I}, \tag{76}$$

which of course are adjoints of each other. Moreover, in this case $[A, A^\dagger] = \hat{I}$, so that the oscillator algebra remains unchanged.

By some similar procedures, we get the corresponding operators,

$$a_{F^{-1}} = \hat{T} a \hat{T}^{-1} = a + \lambda\hat{I} \ , \qquad a_{F^{-1}}^\dagger = \hat{T} a^\dagger \hat{T}^{-1} = G(a)^{-1} a^\dagger G(a) \ , \tag{77}$$

on $\mathfrak{H}_{F^{-1}}$. Once again we obtain $[a_{F^{-1}}, a_{F^{-1}}^\dagger] = \hat{I}$ and similarly for the operator $A' = a_{F^{-1}} = a + \lambda\hat{I}$ and its adjoint $A'^\dagger = a^\dagger + \lambda\hat{I}$ on $\mathfrak{H}$.

Let now define the vectors

$$\phi_n^F = \hat{T}^{-1}\phi_n = e^{\lambda a^\dagger} G(a)\phi_n \ , \tag{78}$$

which form an orthonormal set in $\mathfrak{H}_F$, and build the corresponding canonical CSs

$$\eta_z^F == e^{\lambda a^\dagger} G(a)\eta_z = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\phi_n^F \ , \tag{79}$$

on $\mathfrak{H}_F$. Considering these as vectors in $\mathfrak{H}$, one can see that

$$a\eta_z^F = (z+\lambda)\eta_z^F . \tag{80}$$

Thus, up to a constant factor, η_z^F is just the canonical CSs on $\mathfrak{H}$ corresponding to the point $(z+\lambda) \in \mathbb{C}$ such that the following holds for η_z^F in (79)

$$\eta_z^F = C(\lambda, z) \sum_{n=0}^{\infty} \frac{(z+\lambda)^n}{\sqrt{n!}} \ \phi_n \ ,$$

where the constant $C(\lambda, z)$ can be computed by going back to (79). Thus, one gets $C(\lambda, z) = G(z)e^{-\frac{|z|^2}{2}}$ and finally

$$\eta_z^F = G(z)e^{-\frac{|z|^2}{2}} \ \sum_{n=0}^{\infty} \frac{(z+\lambda)^n}{\sqrt{n!}} \ \phi_n = G(z)e^{\lambda(\Re(z)+\frac{\lambda}{2})}\eta_{z+\lambda} \ . \tag{81}$$

Comparing (81) with (72) and writing $\eta_z^F = \zeta_{z+\lambda}$, it can be found that $w = z+\lambda$, $x_n = n$ and $\psi_n = \phi_n$. Furthermore, $\mathcal{N}'(|w|^2) = e^{|z|^2}|G(z)|^{-2}$ and $\Omega(w) = e^{i\Theta(w)}$, where It is written $G(z) = |G(z)|e^{i\Theta(w)}$. It is remarkable that in this example while η_z^F is written in (79) in terms of a non-orthonormal basis $\{\phi_n^F\}_{n=0}^{\infty}$, when these vectors are considered as constituting a basis for $\mathfrak{H}$, its transcription in terms of the orthonormal basis $\{\phi_n\}_{n=0}^{\infty}$ only involves a shift in the variable z and no change in the components.

It is now straightforward to write down a resolution of identity, following the pattern of the canonical CSs. Indeed, writing $w = z+\lambda = \rho e^{i\theta}$, we have (on $\mathfrak{H}$),

$$\int\!\!\int_{\mathbb{C}} |\zeta_w\rangle\langle\zeta_w|\mathcal{N}'(|w|^2)d\mu(w,\overline{w}) = \hat{I} \ , \qquad d\mu(w,\overline{w}) = \frac{e^{-\rho^2}}{\pi}\rho d\rho d\theta \ . \tag{82}$$

The dual CSs $\eta_z^{F^{-1}}$ are obtained by replacing the ϕ_n^F in (78) by $\phi_n^{F^{-1}} = \hat{T}\phi_n = G(a)^{-1}e^{-\lambda a^\dagger}\phi_n$. But since $G(a)^{-1}e^{-\lambda a^\dagger} = e^{-\lambda a^\dagger}G(a-\lambda\hat{I})^{-1} = e^{-\lambda a^\dagger}G(a-\lambda\hat{I})^{-1}\phi_n$. Hence, using the same argument as with the ϕ_n^F, we will observe that in the present case (up to normalization), the dual pair of states η_z^F and $\eta_z^{F^{-1}}$ is obtained simply by replacing λ by $-\lambda$.

Two particular cases of the operator $\hat{T}^{-1}$ in (74) are of special interest. In the first instance take $G(a) = \hat{I}$, so that $\hat{T}^{-1} = e^{\lambda a^\dagger}$. The vectors $\phi_n^F = \hat{T}^{-1}\phi_n$ may easily be calculated. Indeed we get

$$\phi_n^F = \sum_{k=0}^{\infty} \frac{(\lambda a^\dagger)^k}{k!}\phi_n = e^{\frac{\lambda^2}{2}} \frac{a^{\dagger n}}{\sqrt{n!}}\eta_\lambda \ , \tag{83}$$

which (up to normalization) are the well-known *photon-added CSs* of quantum optics [6, 86]. Hence in this case we write $\phi_n^F = \phi_{\lambda,n}^{\mathrm{pa}}$. We denote the corresponding CSs by $\eta_{\lambda,z}^{\mathrm{pa}}$ and note that

$$\eta_z^F := \eta_{\lambda,z}^{\mathrm{pa}} = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\phi_{\lambda,n}^{\mathrm{pa}} = e^{\lambda(x+\frac{\lambda}{2})}\eta_{z+\lambda} \ , \tag{84}$$

where $x = \Re(z)$. Clearly if $\lambda \longrightarrow 0$, then $\eta^{\mathrm{pa}}_{\lambda,z} \longrightarrow \eta_z$. It ought to be emphasized at this point, however, that while the vectors $\phi^{\mathrm{pa}}_{\lambda,n}$ are (up to normalization) photon added CSs, the vectors $\eta^{\mathrm{pa}}_{\lambda,z}$ are just (up to normalization) canonical CSs. The dual set of CSs, $\eta^{F^{-1}}_z$ are obtained by replacing λ by $-\lambda$ so that the states $\eta^{\mathrm{pa}}_{\lambda,z}$ and $\eta^{\mathrm{pa}}_{-\lambda,z}$, $z \in \mathbb{C}$, are in duality, and the interesting relation,$\langle \eta^{\mathrm{pa}}_{-\lambda,z} | \eta^{\mathrm{pa}}_{\lambda,z} \rangle_{\mathfrak{H}} = e^{-\lambda(\lambda+2iy)}$ holds.

On $\mathfrak{H}_F$ one has the creation and annihilation operators (see (75)),

$$a_F = a - \lambda \hat{I}, \quad a^\dagger_F = a^\dagger \ , \tag{85}$$

which are adjoints of each other on $\mathfrak{H}_F$, but clearly not so on $\mathfrak{H}$. However, on $\mathfrak{H}$ the two operators A and $A^\dagger$ are as in (76):

$$A = a - \lambda \hat{I}, \quad A^\dagger = a^\dagger - \lambda \hat{I} \ .$$

As the second particular case of (74), we take $\lambda = 0$ and $G(a) = e^{\mu a}$, $\mu \in \Re$, i.e., $\hat{T}^{-1} = e^{\mu a}$. The basis vectors are now

$$\phi^F_n = e^{\mu a}\phi_n = \sqrt{n!}\sum_{k=0}^{n} \frac{\mu^{n-k}}{\sqrt{k!}(n-k)!}\ \phi_k = \frac{(a^\dagger + \mu \hat{I})^n}{\sqrt{n!}}\ \phi_0. \tag{86}$$

These states have also been studied in the quantum optical literature [38] and in view of the last expression in (90), we shall call them *binomial states* and write $\phi^F_n = \phi^{\mathrm{bin}}_{\mu,n}$. The CSs, built out of these vectors as basis states, are:

$$\begin{aligned} \eta^F_z := \eta^{\mathrm{bin}}_{\mu,z} = e^{\mu a}\eta_z &= e^{-|z|^2/2}\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\ \phi^{\mathrm{bin}}_{\mu,n} \\ &= e^{\mu x - |z|^2/2}\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\ \phi_n. \end{aligned} \tag{87}$$

The dual family of CSs are simply $\eta^{\mathrm{bin}}_{-\mu,z}$ and $\langle \eta^{\mathrm{bin}}_{-\mu,z} | \eta^{\mathrm{bin}}_{\mu,z} \rangle = 1$. The creation and annihilation operators on $\mathfrak{H}_F$ are:

$$a_F = a \ , \qquad a^\dagger_F = a^\dagger + \lambda \hat{I} \ , \tag{88}$$

while the other two operators on $\mathfrak{H}$ are:

$$A = a \ , \qquad A^\dagger = a^\dagger \ . \tag{89}$$

The operators (88) have been studied, in the context of on-self-adjoint Hamiltonians in [14, 15, 16]. Again, it is remarkable that the CSs $\eta^{\mathrm{bin}}_{\mu,z}$ are exactly the canonical CSs, η_z, up to a factor.

At the end of this subsection, I briefly remind that we have established in Ref. [83] the relation

$$\begin{aligned} \phi_{m,\mu} &= \frac{e^{\mu a}}{\sqrt{L^{(0)}_m(-\mu^2)}}\ \phi_m = \sum_{n=0}^{m} \frac{\sqrt{m!}\ \mu^{m-n}}{(m-n)!\sqrt{n! L^{(0)}_m(-\mu^2)}}\ \phi_n \\ &= \left(\frac{\mu^{2m} m!}{L^{(0)}_m(-\mu^2)} \right)^{1/2} \sum_{n=0}^{m} \frac{(\mu^{-1})^n}{\sqrt{n!}[(m-n)]!}\ \phi_n = |\mu^{-1}, f\rangle, \end{aligned} \tag{90}$$

where $L^{(0)}_m(-\mu^2)$ are the $m-$order Laguerre functions and the states $|\mu^{-1}, f\rangle$ have been demonstrated as the bound state nonlinear CSs. Then we have used the states in (90) as the basis of a $f-$deformed Fock space (with nonlinearity function $f(\hat{n}) = (m - \hat{n})$) and finally the representations of coherent and squeezed states in these new basis together with their statistics have been studied in detail [83].

3.4.2. Re-scaled basis states and nonlinear CSs For the next general class of examples, let the operator $\hat{T}^{-1}$ have the form

$$\hat{T}^{-1} := \hat{T}(\hat{n})^{-1} = \sum_{n=0}^{\infty} \frac{1}{t(n)} |\phi_n\rangle\langle\phi_n|, \tag{91}$$

where $t(n)$s are real numbers, having the properties:

(i) $t(0) = 1$ and $t(n) = t(n')$ if and only if $n = n'$;

(ii) $0 < t(n) < \infty$;

(iii) the finiteness condition for the limit

$$\lim_{n\to\infty} \left[\frac{t(n)}{t(n+1)}\right]^2 \cdot \frac{1}{n+1} = \rho < \infty \tag{92}$$

holds, which implies that the series

$$\sum_{n=0}^{\infty} \frac{r^{2n}}{[t(n)]^2 n!} := S(r^2), \tag{93}$$

converges for all $r < L = 1/\sqrt{\rho}$. The operators $\hat{T}$ and $\hat{F}$ are now

$$\hat{T} := \hat{T}(\hat{n}) = \sum_{n=0}^{\infty} t(n)|\phi_n\rangle\langle\phi_n| \;, \qquad \hat{F} := \hat{F}(\hat{n}) = \sum_{n=0}^{\infty} t(n)^2|\phi_n\rangle\langle\phi_n| \;. \tag{94}$$

Let me define a new operator $f(\hat{n})$, by its action on the basis vectors.

$$f(\hat{n})\phi_n := \frac{t(n)}{t(n-1)}\phi_n = f(n)\phi_n, \tag{95}$$

then

$$t(n) = f(n)f(n-1)\cdots f(1) := [f(n)]!. \tag{96}$$

Thus one has the transformed, non-orthogonal basis vectors

$$\phi_n^F = \frac{1}{t(n)}\phi_n = \frac{1}{[f(n)]!}\phi_n \;, \tag{97}$$

We shall call the vectors (97) *re-scaled basis states*.

The CSs η_z^F are now:

$$\eta_z^F = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n^F, \tag{98}$$

which, as vectors in $\mathfrak{H}_F$ are well defined and normalized for all $z \in \mathbb{C}$. However, when considered as vectors in $\mathfrak{H}$ and rewritten as:

$$\eta_z^F = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{[f(n)]!\sqrt{n!}} \phi_n, \tag{99}$$

are no longer normalized and defined only on the domain (see (92) and (93)),

$$\mathcal{D} = \left\{ z \in \mathbb{C} \,\middle|\, |z| < L = \frac{1}{\rho} \right\}. \tag{100}$$

The operators a_F and $a_F^\dagger$ act on the vectors ϕ_n^F as

$$a_F\phi_n^F = \sqrt{n}\phi_{n-1}^F\ , \qquad a_F^\dagger\phi_n^F = \sqrt{n+1}\phi_{n+1}^F\ . \tag{101}$$

The operator $A = a_F$, considered as an operator on $\mathfrak{H}$ and its adjoint $A^\dagger$ on $\mathfrak{H}$ act on the original basis vectors ϕ_n in the manner,

$$A\phi_n = f(n)\sqrt{n}\phi_{n-1}\ , \qquad A^\dagger\phi_n = f(n+1)\sqrt{n+1}\phi_{n+1}\ , \tag{102}$$

and thus, they can be written in an obvious notation as

$$A = af(\hat{n})\ , \qquad A^\dagger = f(\hat{n})a^\dagger\ , \tag{103}$$

as operators on $\mathfrak{H}$.

Thus, up to a normalization factor, the CSs defined in (99) are the well-known *nonlinear CSs* of quantum optics [61, 64].

The dual CSs $\eta_z^{F^{-1}}$ which, as vectors in the Hilbert space $\mathfrak{H}_{F^{-1}}$, will be well-defined vectors in $\mathfrak{H}$ only if

$$\lim_{n\to\infty}\left[\frac{t(n+1)}{t(n)}\right]^2\cdot\frac{1}{n+1} = \widetilde{\rho} < \infty. \tag{104}$$

From now on, the sign "tilde" over the "operators" and "states", assign them to the corresponding "dual operators" and "dual states", respectively. In this case, one has

$$\eta_z^{F^{-1}} = \widetilde{\eta}\ _z^F = \mathcal{N}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{[f(n)]!z^n}{\sqrt{n!}}\ \phi_n, \tag{105}$$

and are defined (as vectors in $\mathfrak{H}$) on the domain

$$\widetilde{\mathcal{D}} = \left\{z\in\mathbb{C}\Big||z| < \widetilde{L} = \frac{1}{\sqrt{\widetilde{\rho}}}\right\}. \tag{106}$$

Equations (105) and (106) should be compared to (99) and (100). One also has $\langle\eta_z^{F^{-1}}\Big|\eta_z^F\rangle_{\mathfrak{H}} = 1$, for all $z\in\mathcal{D}\cap\widetilde{\mathcal{D}}$.

A resolution of the identity on $\mathfrak{H}$ can be obtained in terms of the vectors η_z^F (or $\eta_z^{F^{-1}}$) by solving the moment condition

$$\int_0^L r^{2n}d\lambda(r) = \frac{[f(n)!]^2\, n!}{2\pi}, \quad n = 0,1,2,\cdots\ . \tag{107}$$

A highly instructive example of the duality between families of nonlinear CSs is provided by the GP CSs $|z,\kappa\rangle_{GP} = \eta_z^{\rm GP}$ introduced in (21) [45] and BG CSs $|z,\kappa\rangle_{BG} = \eta_z^{\rm BG}$ introduced in (20) [13], defined for the discrete series representations of the group $SU(1,1)$. It is now immediately clear that the operator

$$\hat{T}(\hat{n}) = \sum_{n=0}\frac{1}{\sqrt{(2\kappa+n-1)!}}|\phi_n\rangle\langle\ \phi_n|\ , \tag{108}$$

acts in the manner,

$$\eta_z^{\rm BG} = \lambda_1\ \hat{T}(\hat{n})\eta_z \qquad \text{and} \qquad \eta_{\rm z}^{\rm GP} = \lambda_2\ \hat{\rm T}(\hat{\rm n})^{-1}\eta_{\rm z}\ , \tag{109}$$

where λ_1 and λ_2 are constants, thus demonstrating the relation of duality between the two sets of CSs.

4. Introducing the generalized displacement operators and the dual family of MP CSs

After we recasted the KPS, HG, PS, TC and $su(1,1)$ CSs as nonlinear CSs and found a nonlinearity function for each set of them, it is now possible to construct all of them through a displacement type operator formalism. I will do this in two distinct ways.
I) Applying the mathematical physics formalism for each set of the nonlinear CSs have been established in section 3 which obtained by the action of $\hat{T}^{-1}(f(\hat{n}))$ on canonical CSs, we are now ready to introduce the dual family associated with each class of them, using the action of $\hat{T}(f(\hat{n}))$ on the canonical CSs. But this is not the only formalism to find the dual family. In what follows another formalism presented by B Roy and P Roy will be explained [87].
II) According to the proposition in [87] the authors have been defined two new operators:

$$B = a\frac{1}{f(\hat{n})}, \qquad B^\dagger = \frac{1}{f(\hat{n})}a^\dagger, \tag{110}$$

such that the canonical commutation relations $[A, B^\dagger] = 1 = [B, A^\dagger]$ hold. Before we proceed any further, to clarify more the problem, an interesting result may be given here. Choosing special compositions of the operators A, $A^\dagger$ in (3) and B, $B^\dagger$ in (110), we may observe that $B^\dagger A|n\rangle = n|n\rangle = A^\dagger B|n\rangle$. It can easily be investigated that the generators $\{A, B^\dagger, B^\dagger A, \hat{I}\}$ constitute the commutation relations of the Lie algebra h_4. The corresponding Lie group is the well-known Weyl-Heisenberg(W-H) group denoted by H_4. The same situation holds for the set of generators $\{B, A^\dagger, A^\dagger B, \hat{I}\}$.

Coming back again to the Roy and Roy formalism, the relations in (110) allow one to define two generalized displacement operators

$$\widetilde{D}_f(z) = \exp(zA^\dagger - z^* B), \qquad \widetilde{D}_f(z)|0\rangle = |\widetilde{z, f}\rangle, \tag{111}$$

$$D_f(z) = \exp(zB^\dagger - z^* A), \qquad D_f(z)|0\rangle = |z, f\rangle. \tag{112}$$

Noting that $\widetilde{D}_f(z) = D_f(-z)^\dagger = [D_f(z)^{-1}]^\dagger$, it may be realized that the *dual* pairs obtained generally from the actions of (111) and (112) on the vacuum state are the orbits of a projective *nonunitary* representations of the W-H group [5], so we named *displacement type* or *generalized displacement* operator. Therefore (using each one of the above two formalisms) it is possible to construct a new class of CSs related to each set of MP CSs, $|z, f\rangle_{\mathrm{Mp}}$ introduced in section 3.

Example 1 *The dual family of KPS CSs:*

Considering the KPS introduced in [52], one can see that:

$$D_f(z)|0\rangle \equiv |z\rangle_{\mathrm{KPS}} \tag{113}$$

and the other one which is a new family of CSs, named *"dual states"* in [5, 87] is:

$$\widetilde{D}_f(z)|0\rangle \equiv |\widetilde{z}\rangle_{\mathrm{KPS}} = \widetilde{\mathcal{N}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n \sqrt{\rho(n)}}{n!}|n\rangle, \tag{114}$$

where the normalization constant is determined as: $\widetilde{\mathcal{N}}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}\rho(n)}{(n!)^2}$. Obviously the states $|\tilde{z}\rangle_{KPS}$ in (114) are new ones, other than $|z\rangle_{KPS}$. By the same procedures we have done

in the previous sections it may be seen that the new states, $|\tilde{z}\rangle_{KPS}$ also can be considered as nonlinear CSs with the nonlinearity function

$$\tilde{f}_{KPS}(\hat{n}) = \sqrt{\frac{\hat{n}\rho(\hat{n}-1)}{\rho(\hat{n})}}, \tag{115}$$

which is exactly the inverse of $f_{\text{KPS}}(\hat{n})$ derived in (36), as one may expect. Also the Hamiltonian for the dual oscillator is found to be

$$\tilde{\hat{H}}_{\text{KPS}} = \hat{n}\left(\tilde{f}_{\text{KPS}}(\hat{n})\right)^2 = \hat{n}^2 \hat{H}^{-1}_{\text{KPS}}. \tag{116}$$

The above results may also be obtained using the mathematical physics formalism. One can define the operator

$$\hat{T} = \sum_{n=0}^{\infty} \sqrt{\frac{n!}{\rho(n)}} |n\rangle\langle n|, \tag{117}$$

the action of which on canonical CSs, $|z\rangle_{CCS} = \exp\left(-|z|^2/2\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle$, yields the KPS CSs:

$$\hat{T}|z\rangle_{\text{CCS}} = |z\rangle_{\text{KPS}}. \tag{118}$$

The $\hat{T}$ operator we introduced in Eq. (117) is well-defined and the inverse of it can be easily obtained as

$$\hat{T}^{-1} = \sum_{n=0}^{\infty} \sqrt{\frac{\rho(n)}{n!}} |n\rangle\langle n|, \tag{119}$$

by which we may construct the new family of dual states: $\hat{T}^{-1}|z\rangle_{\text{CCS}} \equiv |\tilde{z}\rangle_{\text{KPS}}$, which are just the states have been obtained in (114). So, in what follows each of the two approaches which are more easier have been used for constructing the dual pair of each MP CSs.

Applying the presented formalism on the numerous weight functions $\rho(n)$ of the KPS type in Ref. [52] is now easy, so our intention is not to refer to them explicitly (it may be found in [84], completely). Our reconstruction of these states by the standard definitions, i.e. annihilation operator eigen-states and displacement operator techniques, enriches each set of the above classes of CSs in quantum optics, in the context of each other.

Example 2 *The dual family of HG CSs:*

As another class of generalized CSs, we observed that the HG CSs in (14) can be constructed by starting with the hypergeometric function. Now, going back to these states we apply to the canonical CSs on $\mathfrak{H}$, the operators

$$\begin{aligned} \hat{T} \quad &:= \sum_{n=0}^{\infty} \left[\frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n}\right]^{\frac{1}{2}} \ |\phi_n\rangle\langle\phi_n| \ , \\ \hat{T}^{-1} &:= \sum_{n=0}^{\infty} \left[\frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n}\right]^{-\frac{1}{2}} \ |\phi_n\rangle\langle\phi_n| \ . \end{aligned} \tag{120}$$

The explicit form of the dual states as a new set of again HG CSs are as follows (not of the HG CSs type of [9]):

$$|\widetilde{z;p,q}\rangle = |\alpha_1, \cdots \alpha_p; \beta_1, \cdots \beta_q; z\rangle = {}_p\widetilde{\mathcal{N}}_q(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n \sqrt{{}_p\rho_q(n)}}{n!} \ |n\rangle \tag{121}$$

where ${}_p\rho_q(n)$ are defined by the relation (15) and the normalization factor is determined as

$$ {}_p\widetilde{\mathcal{N}}_q(|z|^2) = {}_pF_q(\beta_1, \ldots, \beta_q\ ; \alpha_1, \ldots, \alpha_p;\ x). \tag{122}$$

It is then immediate that the corresponding families of CSs $\{\eta_z^F\}$ and $\{\eta_z^{F^{-1}}\}$ will be in duality. (Actually, it may be necessary to impose additional restrictions on the α_i and β_i, in order to ensure that the CSs $\{\eta_z^F\}$ and $\{\eta_z^{F^{-1}}\}$, when defined on $\mathfrak{H}$, satisfy a resolution of the identity [9]).

Example 3 *The dual family of PS CSs:*

The dual of the PS states (18) can be easily obtained using the approach in [87]:

$$ |\widetilde{q,z}\rangle_{\rm PS} = \widetilde{\mathcal{N}}(q, |z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{\sqrt{n!}} z^n |n\rangle, \tag{123}$$

where $\widetilde{\mathcal{N}}(q, |z|^2)$ is some normalization constant, may be determined. For this example the proposition in [5] works well. The $\hat{T}$-operator in this case reads:

$$ \hat{T} = \sum_{n=0}^{\infty} q^{\hat{n}(\hat{n}-1)/2} |n\rangle\langle n|, \tag{124}$$

by which one may obtain:

$$ \hat{T}^{-1}|z\rangle_{\rm CCS} \equiv \left(\sum_{n=0}^{\infty} q^{-n(n-1)/2} |n\rangle\langle n| \right) |z\rangle_{\rm CCS} = |\widetilde{z,q}\rangle_{\rm PS}. \tag{125}$$

which is exactly the dual states we obtained in Eq. (123).

Example 4 *The dual family of TC CSs*

The dual family of TC CSs of the first kind, may be obtained by either of the two formalisms. Anyway, the results is as follows:

$$ |\widetilde{z;p}\rangle^{(1)}_{\rm TC} = \widetilde{\mathcal{N}}_p(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \sqrt{d_p(n)}\ |n\rangle. \tag{126}$$

And similarly for the second kind one has:

$$ |\widetilde{z;\lambda,\beta}\rangle^{(2)}_{\rm TC} = \widetilde{\mathcal{N}}_{\lambda,\beta}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \sqrt{d_{\lambda,\beta}(n)}\ |n\rangle. \tag{127}$$

where $\widetilde{\mathcal{N}}_p(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n} d_p(n)}{n!}$ and $\widetilde{\mathcal{N}}_{\lambda,\beta}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n} d_{\lambda,\beta}(n)}{n!}$. The parameters used in these relations are the same as the ones have been explained after Eq. (17).

Example 5 *Generalized displacement operators for BG and GP CSs of $su(1,1)$ Lie algebra:*

As a well-known example we express the dual of BG CSs (of $SU(1,1)$) group. The duality of these states with the so-called GP CSs (of $SU(1,1)$) have already been demonstrated in [5]. The latter states were defined as:

$$ |z,\kappa\rangle_{\rm GP} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \sqrt{\frac{(n+2\kappa-1)!}{n!}} z^n |n\rangle. \tag{128}$$

where $\mathcal{N}(|z|^2)$ is a normalization constant. The nonlinearity function and the Hamiltonian may be written as $f_{\rm GP}(\hat{n}) = f^{-1}_{\rm BG}(\hat{n})$ and $\hat{H} = \hat{n}/(\hat{n}+2\kappa-1)$, respectively. Therefore we have in this case:

$$B|\kappa, n\rangle = \sqrt{\frac{n}{n+2\kappa-1}}|\kappa, n-1\rangle, \tag{129}$$

$$B^\dagger|\kappa, n\rangle = \sqrt{\frac{n+1}{n+2\kappa}}|\kappa, n+1\rangle, \tag{130}$$

$$[B, B^\dagger]|\kappa, n\rangle = \frac{2\kappa-1}{(n+2\kappa)(n+2\kappa-1)}|\kappa, n\rangle. \tag{131}$$

where $B = af_{GP}(\hat{n})$ and $B^\dagger = f_{GP}(\hat{n})a^\dagger$ are the deformed annihilation and creation operators for the dual system (GP CSs), respectively. It is immediately observed that $[A, B^\dagger] = \hat{I}, [B, A^\dagger] = \hat{I}$, where obviously $A = af_{BG}(\hat{n})$ and $A^\dagger$ is its Hermition conjugate. So it is possible to obtain the displacement type operators for the BG and GP (nonlinear) CSs discussed in this tutorial, using relations (111) and (112). As a result the displacement operators obtained by presented method are such that:

$$|z, \kappa\rangle_{\rm BG} = D_{\rm BG}(z)|\kappa, 0\rangle = \exp\left(zA^\dagger - z^* B\right)|\kappa, 0\rangle, \tag{132}$$

and

$$|z, \kappa\rangle_{\rm GP} = D_{\rm GP}(z)|\kappa, 0\rangle = \exp\left(zB^\dagger - z^* A\right)|\kappa, 0\rangle. \tag{133}$$

To apply the procedure to the $su(1,1)$-BG CSs for Landau levels we expressed as another example, one must re-defined the auxiliary operators B and $B^\dagger$, in place of the ones introduced in (110), as follows:

$$B = \frac{1}{f_{\rm LL}(\hat{n})}a, \qquad B^\dagger = a^\dagger \frac{1}{f_{\rm LL}(\hat{n})}. \tag{134}$$

5. The link between GK-CSs and nonlinear CSs

As it has mentioned earlier the GK-CSs can not be fully placed in the above two formalisms. We will pay attention to this matter in the following section. Indeed we have to try some *radically different method* from the previous ones.

5.1. A discussion on the modification of GK-CSs

As it is observed previously in subsection 2.6.1, in the modification imposed by El Kinani and Daoud on the GK-CSs, the parameter α has been *implicitly* considered as a constant, which its presence in the exponential factor of the introduced CSs preserves the temporal stability requirement (it is not now an integration *variable*). Meanwhile, for the temporal stability of the GKCSs in (32) one reads:

$$e^{-i\hat{H}t}|z, \alpha\rangle = |z, \alpha'\rangle, \qquad \alpha' = \alpha + \omega t. \tag{135}$$

Upon a closer inspection, one can see that the latter relation is indeed inconsistent with the resolution of the identity. By this we mean that when α is considered as a constant parameter, it really labels any over-complete set of GKCSs, $\{|z, \alpha\rangle\}$. But the time evolution operator in

(135) maps the over-complete set of states $\{|z,\alpha\rangle\}$ to another over-complete set $\{|z,\alpha'\rangle\}$. These are two *distinct set of CSs*, each labels with a specific α, if one consider the El kinani-Daoud formalism. But the temporal stability precisely means that the time evolution of a CS remains a CS, *of the same family*. So transparently speaking, the states introduced in (32) are not of the Gazeau-Klauder type, exactly.

To overcome this problem, we redefine the resolution of the identity as follows:

$$\lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma}d\alpha\int_0^R |z,\alpha\rangle\langle z,\alpha|d\lambda(z)=\sum_{n=0}^{\infty}|n\rangle\langle n|=\hat{I},\qquad 0<R\leq\infty. \tag{136}$$

One can simplify the LHS of (136) which interestingly led exactly to the LHS of (33). Indeed one gets:

$$\int_0^R |z,\alpha\rangle\langle z,\alpha|d\lambda(z)=\lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma}d\alpha\int_0^R |z,\alpha\rangle\langle z,\alpha|d\lambda(z), \tag{137}$$

where $d\lambda(z)$ is determined as in (34).

By this fact it may be concluded that both of the over-complete collection of states: $\{|z,\alpha\rangle\}$ and $\{|z,\alpha'\rangle\}$, when α and $\alpha'\equiv\alpha+\omega t$ both are *fixed*, belong to a large set of over-complete states with an arbitrary α:

$$\{|z,\alpha\rangle, z\in C, -\infty\leq\alpha\leq\infty\}. \tag{138}$$

Note that by replacing $\alpha\in R$ with $-\infty\leq\alpha\leq\infty$ in (138) we want to emphasis that we relax α from the constraint of being fixed. But unfortunately, the *variability* of α destroys the well definition of the operator $f(\alpha,\hat{n})$ will be introduced later in (139) and therefore deformed annihilation and creation operators A and $A^\dagger$. To overcome this difficulty we may bridge the gap between these two situations: variability and constancy of α. We define the set of operators $A=af(\alpha,\hat{n})$, $A^\dagger=f^\dagger(\alpha,\hat{n})a^\dagger$ and any other operator which explicitly depends on α, in each *sector* (*subspace*) $\mathfrak{H}_\alpha$, labeled by a specific α parameter, of the whole Hilbert space $\mathfrak{H}$ which contains all GKCSs $\{|z,\alpha\rangle\}$. Indeed the whole Hilbert space foliates by each α (remember the continuity of α). Moreover the action of the time evolution operator on any state on a specific sector, falls it down to another sector, both belong to a large Hilbert space. So, *when one deals with the operators that depend on the α parameter, it should necessarily be fixed, while this is not the case when we are dealing with the states.*

5.2. The relation between nonlinear CSs and GKCSs

Following the second formalism presented in this article for the states expressed in (32) one may obtain [84]

$$\begin{aligned} f_{\mathrm{GK}}(\alpha,\hat{n}) &= e^{i\alpha(\hat{e}_n-\hat{e}_{n-1})}\sqrt{\frac{\rho(\hat{n})}{\hat{n}\rho(\hat{n}-1)}},\qquad \alpha \text{ being fixed}\\ &= e^{i\alpha(\hat{e}_n-\hat{e}_{n-1})}\sqrt{\frac{\hat{e}_n}{\hat{n}}}, \end{aligned} \tag{139}$$

where the notation $\hat{e}_n\equiv\rho(\hat{n})/\rho(\hat{n}-1)$ has been choosed.

Moreover, we gain the opportunity to find rising and lowering operators in a safe manner:

$$A_{\mathrm{GK}}=af_{\mathrm{GK}}(\alpha,\hat{n}),\qquad A^\dagger_{\mathrm{GK}}=f^\dagger_{\mathrm{GK}}(\alpha,\hat{n})a^\dagger, \tag{140}$$

where one may easily verify that $A_{\rm GK}|z,\alpha\rangle = z|z,\alpha\rangle$. Obviously the commutation relation between these two ($f-$deformed) ladder operators obeys the relation [61]:

$$\begin{aligned}[A_{\rm GK}, A_{\rm GK}^\dagger] &= \frac{\rho(\hat{n}+1)}{\rho(\hat{n})} - \frac{\rho(\hat{n})}{\rho(\hat{n}-1)} \\ &= \hat{e}_{n+1} - \hat{e}_n. \end{aligned} \tag{141}$$

The special case $\rho(n) = n!$ recovers the standard bosonic commutation relation $[a, a^\dagger] = \hat{I}$. Using the *"normal-ordered"* form of the Hamiltonian as in [84] and taking $\hbar = 1 = \omega$, for the Hamiltonian of GKCSs we get

$$\hat{H}_{\rm GK} \equiv \hat{\mathcal{H}} = A_{\rm GK}^\dagger A_{\rm GK} = \hat{n}\Big|f_{\rm GK}(\alpha,\hat{n})\Big|^2 = \frac{\rho(\hat{n})}{\rho(\hat{n}-1)} = \hat{e_n} \tag{142}$$

shows clearly the independency of the dynamics of the system of α. By the above explanations the deformed annihilation and creation operators $A_{\rm GK}$ and $A_{\rm GK}^\dagger$ of the oscillator algebra, satisfy the eigenvector equations:

$$A_{\rm GK}|n\rangle = \sqrt{e_n}e^{i\alpha(e_n - e_{n-1})}|n-1\rangle, \tag{143}$$

$$A_{\rm GK}^\dagger|n\rangle = \sqrt{e_{n+1}}e^{i\alpha(e_{n+1} - e_n)}|n+1\rangle, \tag{144}$$

$$[A_{\rm GK}, \widetilde{A}_{\rm GK}^\dagger]|n\rangle = (e_{n+1} - e_n)|n\rangle, \tag{145}$$

$$[A_{\rm GK}, \hat{n}] = A_{\rm GK} \qquad [A_{\rm GK}^\dagger, \hat{n}] = -A_{\rm GK}^\dagger. \tag{146}$$

5.3. The dual family of GKCSs as the temporally stable CSs of the dual of KPS CSs

Now, all the necessary tools for introducing the dual family of GKCSs have been prepared. Taking into account the results we observed about the nonlinearity nature of KPS CSs, comparing (2.2) with GK CSs in (32), keeping $\hbar$ and ω in the formulas, one may conclude that:

$$e^{-i\frac{\alpha}{\hbar\omega}\hat{\mathcal{H}}}|z\rangle_{\rm KPS} = |z,\alpha\rangle, \qquad 0 \neq \alpha \in \mathbb{R}, \qquad z \in \mathbb{C}. \tag{147}$$

While $|z\rangle_{\rm KPS}$ states are not temporally stable, $|z,\alpha\rangle$ states enjoy this property.

Now we may outline a relatively evident physical meaning to the arbitrary real α in (32) or (147) as follows: $\alpha \equiv \omega t$, where by t we mean the time that the operator acts on the KPS CSs. It should be mentioned that, in a sense this interpretation has been previously presented for the GKCSs in a compact form in Ref. [7]. But in this tutorial it is outlined in a general framework. If so, then $|z,\alpha\rangle$ can be considered as the evolution of $|z\rangle_{KPS}$. Therefore in a more general framework, one can claim that the action of the evolution type operator

$$\hat{S}(\alpha) = e^{-i\frac{\alpha}{\hbar\omega}\hat{\mathcal{H}}}, \qquad \hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = \hat{I}, \qquad 0 \neq \alpha \in \mathbb{R}, \tag{148}$$

on any non-temporally stable CSs, makes it temporally stable CSs. So, $\hat{S}(\alpha)$ is a nice and novelty operator which falls down any generalized CS to a situation which it restores the temporal stability property. Where we stress on the fact that in (148) the evolved Hamiltonian, $\hat{\mathcal{H}}$ should satisfy $\hat{\mathcal{H}}|n\rangle = \hbar\omega e_n|n\rangle$.

At this point we are ready to find a suitable way to define the dual family of GKCS in a safe manner. First we note that the dual family of KPS CSs introduced in Eq. (10) has already been established in Eq. (114), via the following closed form [84]:

$$|\widetilde{z}\rangle_{\rm KPS} = \widetilde{\mathcal{N}}_{\rm KPS}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\mu(n)}}|n\rangle, \qquad z \in \mathbb{C}, \tag{149}$$

where

$$\mu(n) \equiv \widetilde{\rho}(n) = \frac{(n!)^2}{\rho(n)}, \tag{150}$$

$\mu(n) \equiv \widetilde{\rho}(n)$ is dual correspondence of $\rho(n)$. Equation (150) expresses the relation between the KPS and the associated dual CSs, simply. Obviously $\mathcal{N}_{\text{KPS}}$ and $\widetilde{\mathcal{N}}_{\text{KPS}}$ in (10) and (149) are the normalization constants may be obtained. Therefore, employing the formalism presented in (147) when imposed on "the dual of KPS states" in (149), naturally leads one to the following superposition of Fock space for the *"dual family of GKCSs"* (we refer to as by DGKCS):

$$\begin{aligned} \widetilde{\hat{S}}(\alpha)|\widetilde{z}\rangle_{\text{KPS}} = e^{-i\frac{\alpha}{\hbar\omega}\widetilde{H}}|\widetilde{z}\rangle_{\text{KPS}} &= \widetilde{\mathcal{N}}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n e^{-i\alpha\varepsilon_n}}{\sqrt{\mu(n)}}|n\rangle \\ &= |\widetilde{z,\alpha}\rangle, \qquad z \in \mathbb{C}, \qquad 0 \neq \alpha \in \mathbb{R}, \end{aligned} \tag{151}$$

where $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_{\text{KPS}}$ (because of the unitarity of $\hat{S}(\alpha)$, which preserves the norm), is now determined as:

$$\widetilde{\mathcal{N}}(|z|^2) = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{\mu(n)}. \tag{152}$$

The special case of $\varepsilon_n = n$ will recover the canonical CSs, correctly. Note also that setting $\alpha = \omega t$ in (148) and (151) reduces the operators $\hat{S}(\alpha)$ and $\widetilde{\hat{S}}(\alpha)$ to the well-known *time evolution operators* $\mathcal{U}(t)$ and $\widetilde{\mathcal{U}}(t)$, respectively. The case $\alpha = 0$ in the states in (32) and (151) will recover KPS and the corresponding dual CSs (which certainly are not temporally stable), respectively. The overlap between two states of the dual family of GKCSs takes the following form

$$\langle\widetilde{z,\alpha}|\widetilde{z',\alpha'}\rangle = \widetilde{\mathcal{N}}(|z|^2)^{-1/2}\widetilde{\mathcal{N}}(|z'|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{(z^*z')^n e^{-i\varepsilon_n(-\alpha+\alpha')}}{\mu(n)}\ , \tag{153}$$

which means that the states are essentially nonorthogonal.

In what follows we will observe that the states $|\widetilde{z,\alpha}\rangle$ introduced in (151) are exactly of GK type. It should be noticed that the produced states form a new class of generalized CSs, essentially other than $|z,\alpha\rangle$ in (32). Also it is apparent that for our introduction, we have obtained directly the analytic representation of DGKCS of any arbitrary quantum mechanical system. Again using the nonlinear CSs method proposed in section 2.1, one can now deduce the nonlinearity function for the dual states in Eq. (151) as [84]:

$$\widetilde{f}_{\text{GK}}(\alpha,\hat{n}) = e^{i\alpha(\hat{\varepsilon}_n - \hat{\varepsilon}_{n-1})}\sqrt{\frac{\mu(\hat{n})}{\hat{n}\mu(\hat{n}-1)}}, \qquad \alpha \text{ being fixed}, \tag{154}$$

where the notation $\hat{\varepsilon}_n \equiv \mu(\hat{n})/\mu(\hat{n}-1)$ has been choosed. Therefore the deformed annihilation and creation operators of the dual system in analogue to (142) may expressed explicitly as:

$$\widetilde{A}_{\text{GK}} = a e^{i\alpha(\hat{\varepsilon}_n - \hat{\varepsilon}_{n-1})}\sqrt{\frac{\mu(\hat{n})}{\hat{n}\mu(\hat{n}-1)}}, \tag{155}$$

$$\widetilde{A}^{\dagger}_{\text{GK}} = e^{-i\alpha(\hat{\varepsilon}_n - \hat{\varepsilon}_{n-1})}\sqrt{\frac{\mu(\hat{n})}{\hat{n}\mu(\hat{n}-1)}}a^{\dagger}. \tag{156}$$

The normal-ordered Hamiltonian of dual oscillator in the same manner stated in (142) is:

$$\widetilde{\mathcal{H}}_{\mathrm{GK}} \equiv \hat{\widetilde{\mathcal{H}}} = \widetilde{A}^{\dagger}_{\mathrm{GK}}\widetilde{A}_{\mathrm{GK}} = \frac{\mu(\hat{n})}{\mu(\hat{n}-1)} = \frac{\hat{n}^2}{\hat{e}_n}, \tag{157}$$

which is again independent of α. As a result

$$\hat{\widetilde{\mathcal{H}}}|n\rangle = \widetilde{\mathcal{E}}_n|n\rangle \equiv \hbar\omega\varepsilon_n|n\rangle = \varepsilon_n|n\rangle, \qquad \varepsilon_n \equiv \widetilde{e}_n = \frac{n^2}{e_n}, \tag{158}$$

where again we use the units $\omega = 1 = \hbar$. The right equation in (158) illustrates clearly the relation between the eigenvalues between the two mutual dual systems. The dual family of GKCSs also are required to satisfy the following inequalities:

$$0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n < \varepsilon_{n+1} < \cdots. \tag{159}$$

At this point a question may be arisen, to what extent can we may be sure that the dual family of GKCSs in (151) are of the GK type. It is easy to investigate the four criteria and found the affirmative answer to this question (see for detail Ref. [85]). We only imply the fact that upon the temporal stablity and action identity requirements we are lead to the condition

$$\varepsilon_n = \frac{\mu(n)}{\mu(n-1)}, \tag{160}$$

which by conventional choice of $\mu(0) \equiv 1$, we deduce:

$$\mu(n) = \varepsilon_n\varepsilon_{n-1}...\varepsilon_1 = \Pi_{k=1}^{n}\varepsilon_k \equiv [\varepsilon_n]!\ . \tag{161}$$

So, the nonlinearity function can be expressed in terms of the eigen-values of the Hamiltonian system as follows

$$\widetilde{f}_{\mathrm{GK}}(\alpha, \hat{n}) = e^{i\alpha(\hat{\varepsilon}_n - \hat{\varepsilon}_{n-1})}\sqrt{\frac{\hat{\varepsilon}_n}{\hat{n}}}\ , \qquad \alpha \text{ being fixed.} \tag{162}$$

It should be noted that the same arguments I presented in section 5.1 about the resolution of the identity (and the integration procedures), the α parameter (the states and the operators which depend on it), and the corresponding Hilbert spaces, must also be considered in the dual family of GKCSs built in the present section.

5.4. The introduction of temporally stable or Gazeau-Klauder type of nonlinear CSs

Let me now outline the main idea, in a general framework. It is believed that the property of the temporal stability is intrinsic to the harmonic oscillator and the systems which are unitarily equivalent to it [54]. But in what follows I shall demonstrate that how this important property can be restored by a redefinition of any generalized CSs which can be classified in the nonlinear CSs category. Recall that the nonlinear CSs have been introduced in (7) do not have generally the temporal stability requirement [61]. So, upon adding the results in the previous work [84] and the above explanations, one may go proceed and introduce generally the new notion of *"temporally stable"* or *"Gazeau-Klauder type of nonlinear CSs"* as

$$|z, \alpha\rangle_f = \mathcal{N}_f(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n e^{-i\alpha e_n}}{\sqrt{n!}[f(n)]!}|n\rangle, \qquad e_n = nf^2(n), \qquad 0 \neq \alpha \in \mathbb{R}, \quad z \in \mathbb{C}. \tag{163}$$

We can also define the dual of the latter states by the following expression:

$$|\widetilde{z,\alpha}\rangle_f = \widetilde{\mathcal{N}}_f(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n[f(n)]!e^{-i\alpha\varepsilon_n}}{\sqrt{n!}}|n\rangle, \quad \varepsilon_n = \frac{n}{f^2(n)}, \quad 0 \neq \alpha \in \mathbb{R}, \quad z \in \mathbb{C}, \tag{164}$$

which are indeed the temporally stable version of the nonlinear CSs introduced in [87]. In both of the CSs in (163) and (164), α is a real constant and the normalization factors are independent of α. Setting $\alpha = 0$ in (163) and (164), will recover the old form of Man'ko's and Roy's nonlinear CSs, respectively, which clearly were not temporally stable.

5.4.1. Temporally stable CSs of $SU(1,1)$ group An instructive example of the families of nonlinear CSs is provided by the GP and BG CSs, defined for the discrete series representations of the group $SU(1,1)$. Imposing the proposed formalism on BG CSs in (20), then the *"temporally stable CSs of BG type associated with $SU(1,1)$ group"* can be defined as:

$$|z,\alpha\rangle_{\rm GP}^{SU(1,1)} = \mathcal{N}_{\rm GP}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n e^{-i\alpha \frac{n}{(n+2\kappa-1)}}}{[n!/\Gamma(n+2\kappa)]^{1/2}} |n\rangle, \qquad |z|<1, \tag{165}$$

where $\mathcal{N}_{GP}$ is a normalization factor, may be calculated. Analogously, applying the presented extension on the GP type of CSs in (21), gives immediately *"temporally stable CSs of BG type associated with $SU(1,1)$ group"* as follows:

$$|z,\alpha\rangle_{\rm BG}^{SU(1,1)} \equiv |\widetilde{z,\alpha}\rangle_{\rm GP}^{SU(1,1)} = \mathcal{N}_{\rm BG}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n e^{-i\alpha n(n+2\kappa-1)}}{[n!\Gamma(n+2\kappa)]^{1/2}} |n\rangle, \qquad z \in \mathbb{C}, \tag{166}$$

where once more, $\mathcal{N}_{\rm BG}$ is chosen by normalization of the states.

5.4.2. Temporally stable CSs of PS type and its dual As it is observed, the generalized CSs introduced by Penson and Solomon [76, 84] in Eq. (18) are also nonlinear with $f(n) = q^{(1-n)}$ and therefore the factorized Hamiltonian reads $\hat{\mathcal{H}}_{PS} = \hat{n} q^{2(1-\hat{n})}$. It is stated in [76] that under the action of $\exp(-i\hat{H}t)$ these states are temporally stable, where $\hat{H} = a^\dagger a = \hat{n}$. Knowing that the latter Hamiltonian expresses only the (shifted)quantum harmonic oscillator with the corresponding canonical CS, seemingly to verify the invariance under time evolution operator, it may be more realistic to act the $\exp(-i\hat{\mathcal{H}}_{PS}t) = \exp(-i\hat{n}q^{2(1-\hat{n})}t)$ operator on the states in (18). Clearly by such proposition these states are not temporally stable. Moreover, the presented formalism allows one to construct the temporally stable CSs of PS type as follows

$$|q,z,\alpha\rangle_{\rm PS} \equiv e^{-i\frac{\alpha}{\hbar\omega}\hat{\mathcal{H}}_{\rm PS}} |q,z\rangle = \mathcal{N}(q,|z|^2) \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{\sqrt{n!}} e^{-i\alpha e_n} z^n |n\rangle, \tag{167}$$

where $e_n = nq^{2(1-n)}$, which the requested property may be verified, straightforwardly. We have already introduced the dual of the PS states of Eq. (18). So the temporally stable of the dual states may also be obtained immediately as

$$|\widetilde{q,z,\alpha}\rangle_{\rm PS} \equiv e^{-i\frac{\alpha}{\hbar\omega}\tilde{\hat{\mathcal{H}}}_{\rm PS}} |\widetilde{q,z}\rangle = \widetilde{\mathcal{N}}(q,|z|^2) \sum_{n=0}^{\infty} \frac{q^{\frac{-n(n-1)}{2}}}{\sqrt{n!}} e^{-i\alpha\varepsilon_n} z^n |n\rangle, \tag{168}$$

where $\varepsilon_n = \frac{n}{q^{2(1-n)}}$.

I end this subsection with some remarkable points.

- In the light of the presented explanations the annihilation operator eigenstate for GKCSs and associated dual family are:

$$A_{\rm GK}|z,\alpha\rangle = z|z,\alpha\rangle, \qquad \widetilde{A}_{\rm GK}|\widetilde{z,\alpha}\rangle = z|\widetilde{z,\alpha}\rangle. \tag{169}$$

The deformed annihilation and creation operators $\widetilde{A}_{\rm GK}$ and $\widetilde{A}^\dagger_{\rm GK}$ of the dual oscillator algebra, satisfy the eigenvector equations:

$$\widetilde{A}_{\rm GK}|n\rangle = \sqrt{\varepsilon_n}e^{i\alpha(\varepsilon_n-\varepsilon_{n-1})}|n-1\rangle, \tag{170}$$

$$\widetilde{A}^\dagger_{\rm GK}|n\rangle = \sqrt{\varepsilon_{n+1}}e^{i\alpha(\varepsilon_{n+1}-\varepsilon_n)}|n+1\rangle, \tag{171}$$

$$[\widetilde{A}_{\rm GK}, \widetilde{A}^\dagger_{\rm GK}]|n\rangle = (\varepsilon_{n+1}-\varepsilon_n)|n\rangle, \tag{172}$$

$$[\widetilde{A}_{\rm GK}, \hat{n}] = \widetilde{A}_{\rm GK} \qquad [\widetilde{A}^\dagger_{\rm GK}, \hat{n}] = -\widetilde{A}^\dagger_{\rm GK}. \tag{173}$$

Upon looking on the actions defined in (170) and (171) one can interpret $\widetilde{A}_{\rm GK}$ and $\widetilde{A}^\dagger_{\rm GK}$ as the operators which correctly annihilates and creates one quanta of the deformed photon, respectively. A closer look at the basis of the involved Hilbert space $\mathfrak{H}_\alpha$ in each over-complete set $\{|z,\alpha\rangle\}$, shows that it spanned by the vectors

$$|n,\alpha\rangle = \frac{(\widetilde{A}^\dagger_{\rm GK})^n e^{i\alpha\varepsilon_n}}{\sqrt{[e_n]!}}|0\rangle \equiv |n\rangle, \qquad \widetilde{A}_{\rm GK}|0\rangle = 0. \tag{174}$$

Moreover, the α parameter from the basis has been omitted for simplicity. At last we are able to introduce the generators of the deformed oscillator algebra [20] of Gazeau-Klauder and the corresponding dual family as $\{A_{\rm GK}, A^\dagger_{\rm GK}, \hat{\mathcal{H}}\}$ and $\{\widetilde{A}_{\rm GK}, \widetilde{A}^\dagger_{\rm GK}, \widetilde{\hat{\mathcal{H}}}\}$, respectively.

- The probability distribution for the GKCSs is defined as:

$$\widetilde{\mathbf{P}}(n) = |\langle n|\widetilde{z,\alpha}\rangle|^2 = \widetilde{\mathcal{N}}(|z|^2)^{-1}\frac{|z|^{2n}}{\mu(n)}, \tag{175}$$

which is independent of α parameter.

Let me terminate this section with recalling that there exist also a set of equations such as (170-173) related to GKCSs, may be obtained just by replacing: $\widetilde{A}_{\rm GK}$, $\widetilde{A}^\dagger_{\rm GK}$ and ε_n with $A_{\rm GK}, A^\dagger_{\rm GK}$ and e_n, respectively. The latter have been already derived by applying SUSYQM techniques [104], but re-derivation of them are very easy by our formalism. According to their results, the one-dimensional SUSYQM provides a mathematical tool to define ladder operators for an exactly solvable potentials [34]. But the authors did not implied the *explicit* form of the ladder operators, and only the concerning actions were expressed there. Therefore besides the simplicity of our method, it is more complete in the sense that as we observed the *explicit* form of the rising and lowering operators in terms of the the standard bosonic creation and annihilation operators and the photon number (intensity of the field) have been found easily(see equations: (140), (155), (156)).

5.5. Displacement operators associated with GKCS and the corresponding dual family

After which the explicit form of the deformed annihilation operator (and hence the annihilation operator definitions for GKCSs and associated dual family according to Eqs. in (169)) have been introduced, now we are in the position to extract the CSs of Klauder-Perelomov type for an arbitrary quantum mechanical system. For this purpose let introduce the auxiliary operators related to GKCSs:

$$B_{\mathrm{GK}} = a\frac{1}{f_{\mathrm{GK}}(-\alpha,\hat{n})}, \qquad B^{\dagger}_{\mathrm{GK}} = \frac{1}{f^{\dagger}_{\mathrm{GK}}(-\alpha,\hat{n})}a^{\dagger}. \tag{176}$$

and analogously those for the dual families of GKCSs:

$$\widetilde{B}_{\mathrm{GK}} = a\frac{1}{\widetilde{f}(-\alpha,\hat{n})}, \qquad \widetilde{B}^{\dagger}_{\mathrm{GK}} = \frac{1}{\widetilde{f}(-\alpha,\hat{n})}a^{\dagger}. \tag{177}$$

Notice that the minus sign in $(-\alpha)$ of the argument of the f and $\widetilde{f}$-functions is needed in both cases, since only in such cases we have $f^{\dagger}_{\mathrm{GK}}(-\alpha,\hat{n}) = f_{\mathrm{GK}}(\alpha,\hat{n})$.

The constructed f-deformed operators in (176) and (177) are canonically conjugate of the f-deformed creation and annihilation operators $(A_{\mathrm{GK}}, A^{\dagger}_{\mathrm{GK}})$ and $(\widetilde{A}_{\mathrm{GK}}, \widetilde{A}^{\dagger}_{\mathrm{GK}})$, respectively; i.e. satisfy the algebras $[A_{\mathrm{GK}}, B^{\dagger}_{\mathrm{GK}}] = [B_{\mathrm{GK}}, A^{\dagger}_{\mathrm{GK}}] = \hat{I}$ and $[\widetilde{A}_{\mathrm{GK}}, \widetilde{B}^{\dagger}_{\mathrm{GK}}] = [\widetilde{B}_{\mathrm{GK}}, \widetilde{A}^{\dagger}_{\mathrm{GK}}] = \hat{I}$, respectively. Now we have all mathematical instruments to construct the displacement operators for GKCSs:

$$D_{\mathrm{GK}}(z,\alpha) = \exp(zB^{\dagger}_{\mathrm{GK}} - z^{*}A_{\mathrm{GK}}) \tag{178}$$

and for the dual family of GKCSs in a similar manner:

$$\widetilde{D}_{\mathrm{GK}}(z,\alpha) = \exp(z\widetilde{B}^{\dagger}_{\mathrm{GK}} - z^{*}\widetilde{A}_{\mathrm{GK}}). \tag{179}$$

The actions of $D_{\mathrm{GK}}(z,\alpha)$ and $\widetilde{D}_{\mathrm{GK}}(z,\alpha)$ in the latter equations on the vacuum state $|0\rangle$ yield the GKCSs and the associated dual family, up to some normalization constant,respectively, as we demand. From the group theoretical point of view, one can see that the sets $\{A_{\mathrm{GK}}, B^{\dagger}_{\mathrm{GK}}, B^{\dagger}_{\mathrm{GK}}A_{\mathrm{GK}}, \hat{I}\}$ and $\{\widetilde{A}_{\mathrm{GK}}, \widetilde{B}^{\dagger}_{\mathrm{GK}}, \widetilde{B}^{\dagger}_{\mathrm{GK}}\widetilde{A}_{\mathrm{GK}}, \hat{I}\}$ which are respectively responsible for GKCSs and dual family of GKCSs, form the Lie algebra h_4, and the corresponding Lie group is the well-known Weyl-Heisenberg (WH) group. Also the action of the latter operators on the vacuum are the orbits of the projective *non-unitary* representations of the W-H group [5]. It must be understood that as it is pointed out earlier, neither the formalism in [5] nor the equivalent formalism of Ref. [87] for constructing the dual states have not been applied, since the states obtained from the earlier formalisms were not full consistent with the Gazeau-Klauder's criteria. Indeed a rather new way has proposed in this tutorial, based on the central idea, through viewing the GKCSs $|z,\alpha\rangle$ and its dual pair $|\widetilde{z,\alpha}\rangle$ as generalization of KPS nonlinear CSs $|z\rangle_{\mathrm{KPS}}$ and its dual $|\widetilde{z}\rangle_{\mathrm{KPS}}$ to the two distinct temporally stable CSs, respectively. Speaking otherwise, the operators introduced in (178) and (179) do not have the relation: $\widetilde{D}_{GK}(-z,\alpha) = D_{GK}(z,\alpha) = [D_{GK}(z,\alpha)^{-1}]^{\dagger}$, which is the characteristics of the earlier formalisms. Also it is possible to build the following displacement type operator, $V_{\mathrm{GK}}(z,\alpha) = \exp(zA^{\dagger}_{\mathrm{GK}} - z^{*}B_{\mathrm{GK}})$ for GKCSs, and in a similar manner, $\widetilde{V}_{\mathrm{GK}}(z,\alpha) = \exp(z\widetilde{A}^{\dagger}_{\mathrm{GK}} - z^{*}\widetilde{B}_{\mathrm{GK}})$ for dual family of GKCSs, the actions of which on the vacuum state, yield two new sets of states. But it is easy to check that none of them can be classified in the Gazeau-Klauder CSs.

By various superpositions of CSs, different nonclassical states of light may be constructed. Recently, there has been much interest in the construction as well as generation of these states,

because of their properties in the context of quantum optics. Their different characteristics is due to the various quantum interference between summands.

By various superpositions of CSs, different nonclassical states may be constructed. As an example, the even and odd CSs of canonical CSs as well as other classes of generalized CSs such as nonlinear CSs well studied in the literature [57], due to their nonclassical features, such as squeezing, sub-Poissonian statistics (antibunching) and oscillatory number distribution. The symmetric (antisymmetric) combinations of GKCSs have been introduced in [31]. Similarly using the introduced dual families of GKCSs, one led to the even (+) and odd (−) CSs denoted by $|\widetilde{z,\alpha}\rangle_{\pm}$ [85]. The resulted states (±) satisfy the eigenvalue equations$(\widetilde{A}_{GK})^2|\widetilde{z,\alpha}\rangle_{\pm} = z^2|\widetilde{z,\alpha}\rangle_{\pm}$. The real and imaginary Schrödinger cat states may also obtained by adding and subtracting $|\widetilde{z,\alpha}\rangle$ and $|\widetilde{z^*,\alpha}\rangle$ (z^* is the complex conjugate of z), respectively (the explicit form of them can be seen in Ref. [85]).

5.6. Some physical appearances of the dual family of GKCSs

In order to illustrate the presented idea in this paper, let me apply the formalism on some physical examples which the associated GKCSs have already been known.

Example 1 *Harmonic oscillator:*

As the simplest example one can apply the formalism to the harmonic oscillator Hamiltonian, whose the nonlinearity function is equal to 1, hence $\varepsilon_n = n = e_n$ which results the moments as $\mu(n) = n! = \rho(n)$. Note that we have considered a shifted Hamiltonian to lower the zero-point energy to zero ($e_0 = 0 = \varepsilon_0$). Eventually

$$|\widetilde{z,\alpha}\rangle_{\mathrm{CCS}} = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n e^{-i\alpha n}}{\sqrt{n!}}|n\rangle = |z,\alpha\rangle_{\mathrm{CCS}}, \tag{180}$$

ensures the self-duality of canonical CS. For this example all the Gazeau-Klauder's requirements are satisfied, trivially.

Example 2 *Pöschl-Teller potential:*

Interesting to this potential and its CSs is due to various application in many fields of physics such as atomic and molecular physics. The usual GKCSs for the Pöschl-Teller potential, have been demonstrated nicely by J-P Antoine *etal* [7]. Their obtained results are as follows

$$e_n = n(n+\nu), \qquad \rho(n) = \frac{n!\Gamma(n+\nu+1)}{\Gamma(\nu+1)}, \qquad \nu > 2, \tag{181}$$

with the radius of convergence $R = \infty$. Consequently substitution of the quantities of equation (181), in (150) and (158) one can construct the dual of the already known CSs for Pöschl-Teller potential as

$$|\widetilde{z,\alpha}\rangle_{\mathrm{PT}} = (1-|z|^2)^{(1+\nu)/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\frac{n!\Gamma(\nu+1)}{\Gamma(n+\nu+1)}}}e^{-i\alpha\frac{n}{n+\nu}}|n\rangle, \tag{182}$$

whose radius of convergence is determined as the open unit disk. The overlap between two of these states when $\alpha = \alpha'$ is obtained from(153) as:

$${}_{\mathrm{PT}}\langle\widetilde{z,\alpha}|\widetilde{z',\alpha}\rangle_{\mathrm{PT}} = \left[(1-|z|^2)(1-|z'|^2)\right]^{(1+\nu)/2}(1-zz')^{(-1-\nu)}. \tag{183}$$

To be ensure, for these dual states only investigation of the resolution of the identity has been done, since the other three requirements satisfied obviously. As required one has to find $\varrho(x)$ such that the moment integral

$$\int_0^1 x^n \varrho(x) dx = \frac{n!\Gamma(\nu+1)}{\Gamma(n+\nu+1)} \tag{184}$$

holds. It may be checked that the proper weight function is determined as $\nu(1-x)^{\nu-1}$.

At this point, recall that (181) denotes the eigenvalues of different Hamiltonians. The role of the characteristics of the dynamical system plays by the parameter ν. For instance it is precisely the eigenvalues of the anharmonic (nonlinear) oscillator also, well studied in literature. GKCSs (and GK-CSs) have been discussed in Refs. [31] and [88] in details, respectively. In current example the parameter ν is related to two other parameters, namely λ and κ through the relation $\nu = \lambda + \kappa$, which determine the height and the dept of the well potential. While when one deals with the nonlinear oscillator it has another meaning; e.g. we refer to Ref. [88], in which the interest was due to its usefulness in the study of laser light propagation in a *nonlinear Kerr medium.* In particular ν in the latter case is related to the nonlinear susceptibility of the medium. So, the obtained result in (182) can be exactly used for the anharmonic oscillator, too. To this end, in the next example we see that the case of $\nu = 2$ in (181) is the exact eigenvalues of the infinite potential well.

Example 3 *Infinite well potential:*

The GKCSs for the infinite well, well established by in [7], noting that

$$e_n = n(n+2), \qquad \rho(n) = \frac{n!(n+2)!}{2}, \tag{185}$$

with the radius of convergence $R = \infty$. Consequently inserting (185) in (150) and (158) one can construct the dual of these states as

$$|\widetilde{z,\alpha}\rangle_{\rm IW} = (1-|z|^2)^{3/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\left(\frac{2}{(n+1)(n+2)}\right)}} e^{-i\alpha(\frac{n}{n+2})}|n\rangle, \tag{186}$$

with radius of convergence determined as the open unit disk. Again the overlap between two of these states for the special case $\alpha = \alpha'$ is obtained from(153) as:

$${}_{\rm IW}\langle\widetilde{z,\alpha}|\widetilde{z',\alpha}\rangle_{\rm IW} = -\left[\frac{1}{(-1+|z|^2)(-1+|z'|^2)}\right]^{-3/2} \frac{1}{(zz'-1)}. \tag{187}$$

To clarify the fact that these dual states are actually CSs, we only investigate the resolution of the identity, since the other three requirements satisfied straightforwardly. For this condition we have to find $\varrho(x)$ such that the integral

$$\int_0^1 x^n \varrho(x) dx = \frac{2}{(n+1)(n+2)} \tag{188}$$

holds. It is easy to verify that $\varrho(x) = 2(1-x)$ is the one needs in this case.

Example 4 *Hydrogen-like spectrum:*

We now choose the Hydrogen-like spectrum whose the corresponding CSs, has been a long-standing subject and discussed frequently in the literature. For instance in Refs. [40, 53] the

one-dimensional model of such a system with the Hamiltonian $\hat{H} = -\omega/(\hat{n}+1)^2$ and the eigen-values $E_n = -\omega/(n+1)^2$ has been considered ($\omega = me^4/2$, and $n = 0, 1, 2, \cdots$). But to be consistent with the GKCSs, as it has been done in [53], the energy-levels should be shifted by a constant amount, such that after taking $\omega = 1$ one has the eigen-values e_n and therefore the functions $\rho(n)$ as follows

$$e_n = 1 - \frac{1}{(n+1)^2}, \qquad \rho(n) = \frac{(n+2)}{2(n+1)}, \tag{189}$$

with unit disk centered at the origin as the region of convergence, i.e. $R = 1$. Therefore the related dual family of GKCSs for bound state portion of the Hydrogen-like atom can be constructed. For this purpose, take into account (189) in (150) and (158), so the explicit form of the corresponding dual family of GKCSs for this system can be easily obtained as

$$|\widetilde{z,\alpha}\rangle_{\mathrm{H}} = \left(\frac{1}{2\sqrt{|z|^2}}\left[2I_1(2\sqrt{|z|^2}) + \sqrt{|z|^2}I_2(2\sqrt{|z|^2})\right]\right)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\frac{2n!(n+1)!}{n+2}}} e^{-i\alpha\left(\frac{n(n+1)^2}{n+2}\right)}|n\rangle, \tag{190}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. In this case $\tilde{R} = \infty$ as the radius of convergence. Similar to the preceding examples, we only verify the resolution of the identity. In the present case we have to find a function $\tilde{\sigma}(x)$ such that

$$\int_0^\infty x^n \tilde{\sigma}(x)dx = 2\frac{n!(n+1)!}{n+2}. \tag{191}$$

The integral in (199) is again helpful, if we rewrite the RHS of the (191) as $2n![(n+1)!]^2/(n+2)!$. The suitable measure is then found to be

$$\tilde{\sigma}(x) = G^{3,1}_{0,0}\left(x\middle|\begin{array}{c} 0,1,1, \quad . \\ 2, \quad . \end{array}\right). \tag{192}$$

The overlap between these states for the special case $\alpha = \alpha'$ is obtained from (153) in the closed form

$${}_{\mathrm{H}}\langle\widetilde{z,\alpha}|\widetilde{z',\alpha}\rangle_{\mathrm{H}} = \widetilde{\mathcal{N}}(|z|^2)^{-1/2}\widetilde{\mathcal{N}}(|z'|^2)^{-1/2}\frac{1}{2\sqrt{z^*z'}}\left(2I_1(2\sqrt{z^*z'}) + \sqrt{z^*z'}I_2(2\sqrt{z^*z'})\right), \tag{193}$$

where $\widetilde{\mathcal{N}}(|z|^2) = \frac{1}{2\sqrt{|z|^2}}\left[2I_1(2\sqrt{|z|^2}) + \sqrt{|z|^2}I_2(2\sqrt{|z|^2})\right]$.

Example 5 *Morse potential:*

As the final physical example, I pay attention to the GKCSs for the Morse potential, which is the simplest anharmonic oscillator, useful in various problems in different fields of physics(for example: spectroscopy, diatomic and polyatomic molecule vibrations and scattering), can be obtained using the related quantities may be found in [80]:

$$e_n = \frac{n(M+1-n)}{M+2}, \qquad \rho(n) = \frac{\Gamma(n+1)\Gamma(M+1)}{(M+2)^n\Gamma(M+1-n)}, \tag{194}$$

where $n = 0, 1, \cdots < (M+1)$. Therefore, taking into account (194) in (150) and (158), the dual of these states can be produced as

$$|\widetilde{z,\alpha}\rangle_{\mathrm{MP}} = \left[1 + \frac{|z|^2}{2+M}\right]^{-M/2} \sum_{n=0}^{M} \frac{z^n}{\sqrt{\frac{(M+2)^n\Gamma(n+1)\Gamma(M-n+1)}{\Gamma(M+1)}}} e^{-i\alpha\left(\frac{n(M+2)}{M-n+1}\right)}|n\rangle, \tag{195}$$

where again the equation (152) have been used. Noticing that the series led to the above $\widetilde{\mathcal{N}}(|z|^2)$ is now a finite series, makes it clear that it converges for every values of $x = \sqrt{|z|^2} \geq 0$, i.e. $z \in \mathbb{C}$. For the overlap between two of these states when $\alpha = \alpha'$ the formula (153) is not useful and one must calculate especially the overlap between the Morse states for themselves, because of the upper bound of the evolved sigma:

$$\begin{aligned} {}_{\mathrm{MP}}\langle \widetilde{z,\alpha} | \widetilde{z',\alpha} \rangle_{\mathrm{MP}} &= \widetilde{\mathcal{N}}(|z|^2)^{-1/2} \widetilde{\mathcal{N}}(|z'|^2)^{-1/2} \sum_{n=0}^{M+1} \frac{(z^* z')^n}{\mu(n)} \\ &= \left[\left(1 + \frac{|z|^2}{2+M}\right) \left(1 + \frac{|z'|^2}{2+M}\right) \right]^{-M/2} \left[\frac{2 + M + zz'}{2+M} \right]^M . \end{aligned} \tag{196}$$

Verifying the resolution of the identity is necessary. As before, the function $\varrho(x)$ must be found such that:

$$\int_0^\infty x^n \varrho(x) dx = \frac{(M+2)^n \Gamma(n+1) \Gamma(M-n+1)}{\Gamma(M+1)}. \tag{197}$$

Using the definition of Meijer's G-function and the inverse Mellin theorem, it follows that [63]:

$$\begin{aligned} &\int_0^\infty dx x^{s-1} G^{m,n}_{p,q} \left(\alpha x \middle| \begin{array}{cccccc} a_1, & \cdots, & a_n, & a_{n+1}, & \cdots, & a_p \\ b_1, & \cdots, & b_m, & b_{m+1}, & \cdots, & b_q \end{array} \right) \\ &= \frac{1}{\alpha^s} \frac{\Pi_{j=1}^m \Gamma(b_j + s) \Pi_{j=1}^n \Gamma(1 - a_j - s)}{\Pi_{j=n+1}^p \Gamma(a_j + s) \Pi_{j=m+1}^q \Gamma(1 - b_j - s)}. \end{aligned} \tag{198}$$

One can find the function $\varrho(x)$ needed in (197), in terms of the Meijer's G-function by the expression:

$$\varrho(x) = (M+2)\Gamma(M+1) G^{1,1}_{0,0} \left(x(M+2)^{-1} \middle| \begin{array}{cc} -(M+1), & . \\ 0, & . \end{array} \right). \tag{199}$$

A point is worth to mention in relation to the latter example, which is special in the sense that its Fock space is finite dimension. While in the infinite dimensional cases the presented formalism provide the ladder operators associated with each system, this is not so in the finite dimensional case. Because, in general it is not possible to find the algebraic definition for the CSs defined in the finite dimension. Anyway, Gazeau-Klauder definition provides a suitable and simple rule to define the CSs associated to such quantum mechanical systems, and the presented formalism in this paper provides a safe way to build their dual family.

6. *Introducing the generalized GKCSs and the associated dual family*

In the light of the above explanations we are now in a position to propose the generalized GKCSs, by which we may recover the GKCSs in Eq. (32) and the nonlinear CSs in Eq. (7) as two special cases ‡. In the following scheme, the physical meaning of the α parameter which enters in the GKCSs will be more clear, the case it has already mentioned as $\alpha = \omega t$.

‡ I must thanks to Prof. S Twareque Ali for adding this part to our works

6.1. Time evolved CSs as the generalized GKCSs

Consider the Hamiltonian $\hat{H}$ whose eigenvectors are $|\phi_n\rangle$ and eigenvalues are e_n, such that:

$$\hat{H} = \omega \sum_{n=0}^{\infty} e_n |\phi_n\rangle\langle\phi_n|, \qquad \text{where} \quad \mathrm{\hat{H}}|\phi_\mathrm{n}\rangle = \omega \mathrm{e_n}|\phi_\mathrm{n}\rangle, \tag{200}$$

where ω is a constant with the dimension of energy(taking $\hbar = 1$). Let $\mathfrak{H}$ be a separable, infinite dimensional and complex Hilbert space which spanned by orthonormal set $\{|\phi_n\rangle\}_{n=0}^{\infty}$. Also suppose $0 = e_0 < e_1 < e_2 < \cdots < e_n < e_{n+1} < \cdots$, be such that the sum $\sum_{n=0}^{\infty} \frac{x^n}{[e_n]!}$ converges in some interval $0 < x \leq L$. For $z \in \mathbb{C}$, such that $|z|^2 < L \leq \infty$, define the generalized CSs:

$$|z\rangle \doteq \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[e_n]!}} |\phi_n\rangle, \tag{201}$$

where $\mathcal{N}(|z|^2)$ being a normalization factor. As it is clear these states known as nonlinear CSs, with the nonlinearity function $f(n) = \sqrt{\frac{e_n}{n}}$. Setting $z = re^{i\theta}$ with $r = J^{\frac{1}{2}}$, it is reasonable to write $|z\rangle \equiv |J, \theta\rangle$. Now if $d\nu$ be a measure which solves the moment problem

$$\int_0^L J^n d\nu(J) = [e_n]!, \qquad \int_0^L d\nu(J) = 1, \tag{202}$$

then these CSs satisfy the resolution of the identity

$$\int_0^L \left[\int_0^{2\pi} |J,\theta\rangle\langle J,\theta| \mathcal{N}(J) \frac{d\theta}{2\pi}\right] d\nu(J) = I_{\mathfrak{H}}. \tag{203}$$

The CSs in (201) evolve with time in the manner

$$|z,t\rangle = e^{-i\hat{H}t}|z\rangle = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n e^{-i\omega e_n t}}{\sqrt{[e_n]!}} |\phi_n\rangle, \tag{204}$$

or equivalently in terms of the new variables J and θ

$$|J,\theta,t\rangle = e^{-i\hat{H}t}|J,\theta\rangle = \mathcal{N}(J)^{-1/2} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{in\theta} e^{-i\omega e_n t}}{\sqrt{[e_n]!}} |\phi_n\rangle. \tag{205}$$

This larger set of GKCSs, we will call them *"generalized GKCSs"*, defined for all t, satisfies the resolution of the identity,

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\int_0^L \left\{\int_0^{2\pi} |J,\theta,t\rangle\langle J,\theta,t| \mathcal{N}(J) \frac{d\theta}{2\pi}\right\} d\nu(J)\right] d\mu_{\mathcal{B}} \\
&= \int_0^L \left[\int_0^{2\pi} |J,\theta,t\rangle\langle J,\theta,t| \mathcal{N}(J) \frac{d\theta}{2\pi}\right] d\nu(J) \\
&= \int_{\mathbb{R}} \left[\int_0^L |J,\theta,t\rangle\langle J,\theta,t| \mathcal{N}(J) d\nu(J)\right] d\mu_{\mathcal{B}} \\
&= \sum_{n=0}^{\infty} |\phi_n\rangle\langle\phi_n| = I_{\mathfrak{H}},
\end{aligned} \tag{206}$$

where $d\mu_{\mathcal{B}}$ which is really a functional(not a measure) is referred to as the *Bohr measure*,

$$\langle \mu_{\mathcal{B}}; f\rangle \doteq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)dx = \int_{\mathbb{R}} f(x) d\mu_{\mathcal{B}}(x), \tag{207}$$

and f is a suitably chosen function over $\mathbb{R}$. In particular, if $f(x) = 1$ for all x, then $\langle \mu_{\mathcal{B}}; f\rangle = 1$, so that $\mu_{\mathcal{B}}$ resembles a probability measure. Therefore writing the Bohr measure as an integral only has a symbolic meaning.

Setting $t = 0$ in the "generalized GKCSs" of Eq. (204), we will recover the nonlinear CSs and if $\theta = 0$ in the generalized CSs of Eq. (205) reduces to the GKCSs $|J, \alpha\rangle$ in (25),replacing γ with α, with $\alpha \equiv \omega t$, which the latter states satisfy the resolution of the identity,

$$\int_{\mathbb{R}} \left[\int_0^L |J, \alpha\rangle\langle J, \alpha|\mathcal{N}(J) d\nu(J) \right] d\mu_{\mathcal{B}}(t) = I_{\mathfrak{H}}. \tag{208}$$

The generalized GKCSs $|J, \theta, t\rangle$ in (205), satisfy the stability condition and the action identity, as well as the continuity in the labels and the resolution of the identity,

$$e^{-i\hat{H}t'}|J, \theta, t\rangle = |J, \theta, t + t'\rangle, \qquad \langle J, \theta, t|\hat{H}|J, \theta, t\rangle = \omega J, \tag{209}$$

and so do the states $|z, t\rangle$ in (204).

6.2. *The dual family of the "generalized GKCSs"*

Let me now write $e_n = nf^2(n)$, so using our previous results in the present paper, there are a *dual set* of numbers $\widetilde{e}_n \equiv \varepsilon_n = \frac{n}{f^2(n)}$, associated to the *dual Hamiltonian* $\widetilde{\hat{H}}$. Correspondingly this Hamiltonian has eigenvectors $|\phi_n\rangle$ and eigenvalues ε_n, such that:

$$\widetilde{\hat{H}} = \omega \sum_{n=0}^{\infty} \varepsilon_n |\phi_n\rangle\langle\phi_n|, \qquad \text{where} \quad \widetilde{\hat{H}}|\phi_n\rangle = \omega\varepsilon_n|\phi_n\rangle. \tag{210}$$

Also assuming that $0 = \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n < \varepsilon_{n+1} < \cdots$, be such that the sum $\sum_{n=0}^{\infty} \frac{x^n}{[\varepsilon_n]!}$ converges in some interval $0 < x \leq \widetilde{L}$. We can now define the *dual family* of CSs as in (201) by

$$|\widetilde{z}\rangle \doteq \widetilde{\mathcal{N}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[\varepsilon_n]!}} |\phi_n\rangle, \tag{211}$$

which are the well known(dual) nonlinear CSs of Ref. [87]. The time evolution of these states are as,

$$|\widetilde{z, t}\rangle = e^{-i\hat{H}t}|\widetilde{z}\rangle = \widetilde{\mathcal{N}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n e^{-i\omega\varepsilon_n t}}{\sqrt{[\varepsilon_n]!}} |\phi_n\rangle. \tag{212}$$

Again setting $z = re^{i\theta}$ with $r = J^{\frac{1}{2}}$, we can write $|\widetilde{z}\rangle \equiv |\widetilde{J, \theta}\rangle$. So equivalently the states in (212) can be rewritten in terms of the new variables J and θ as

$$|\widetilde{J, \theta, t}\rangle = e^{-i\hat{H}t}|\widetilde{J, \theta}\rangle = \widetilde{\mathcal{N}}(J)^{-1/2} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{in\theta} e^{-i\omega\varepsilon_n t}}{\sqrt{[\varepsilon_n]!}} |\phi_n\rangle. \tag{213}$$

We call this large set of states as the *"dual of the generalized GKCS"*. Setting $\theta = 0$ in (213) will reduce it to the dual of the GKCSs we introduced in (151) with $\alpha = \omega t$. Provided that the moment problem

$$\int_0^{\widetilde{L}} J^n d\widetilde{\nu}(J) = [\varepsilon_n]!, \qquad \int_0^{\widetilde{L}} d\widetilde{\nu}(J) = 1, \tag{214}$$

has a solution, we also have expressions for the resolution of the identity of the type (203), (206) and (208). The GK criteria may immediately be verified for the dual of the generalized GKCS in Eq. (213), as it was down for the "generalized GKCSs" in Eq. (205).

6.3. Generalized f-deformed creation and annihilation operators

Define the two set of the generalized annihilation operators

$$A|\phi_n\rangle = \sqrt{e_n}|\phi_{n-1}\rangle, \qquad \widetilde{A}|\phi_n\rangle = \sqrt{\varepsilon_n}|\phi_{n-1}\rangle, \tag{215}$$

and the corresponding generalized creation operators

$$A^\dagger|\phi_n\rangle = \sqrt{e_{n+1}}|\phi_{n+1}\rangle, \qquad \widetilde{A}^\dagger|\phi_n\rangle = \sqrt{\varepsilon_{n+1}}|\phi_{n+1}\rangle, \tag{216}$$

where we recall that $\varepsilon_n \equiv \widetilde{e}_n$. So that the Hamiltonian and the associated dual are

$$\hat{H} = \omega A^\dagger A, \qquad \widetilde{\hat{H}} = \omega \widetilde{A}^\dagger \widetilde{A}. \tag{217}$$

For the generalized GKCSs we have

$$A_{\rm GK}|\phi_n\rangle = \sqrt{e_n}e^{i\omega t(e_n - e_{n-1})}|\phi_{n-1}\rangle \tag{218}$$

$$A^\dagger_{\rm GK}|\phi_n\rangle = \sqrt{e_{n+1}}e^{i\omega t(e_{n+1} - e_n)}|\phi_{n+1}\rangle \tag{219}$$

Similarly for the dual family of generalized GKCSs the following holds

$$\widetilde{A}_{\rm GK}|\phi_n\rangle = \sqrt{\varepsilon_n}e^{i\omega t(\varepsilon_n - \varepsilon_{n-1})}|\phi_{n-1}\rangle \tag{220}$$

$$\widetilde{A}^\dagger_{\rm GK}|\phi_n\rangle = \sqrt{\varepsilon_{n+1}}e^{i\omega t(\varepsilon_{n+1} - \varepsilon_n)}|\phi_{n+1}\rangle \tag{221}$$

Then, for the states in (201) and (211) we have

$$A|z\rangle = z|z\rangle, \qquad \widetilde{A}|\widetilde{z}\rangle = z|\widetilde{z}\rangle, \tag{222}$$

and for the states in (204) and (212) we have clearly

$$A_{\rm GK}|z,t\rangle = z|z,t\rangle, \qquad \widetilde{A}_{\rm GK}|\widetilde{z,t}\rangle = z|\widetilde{z,t}\rangle, \tag{223}$$

Using our first formalism (nonlinear CSs method) for reproducing the states $|z,t\rangle$ in (204) and $|\widetilde{z,t}\rangle$ in (212), we introduce the corresponding nonlinearity functions explicitly as

$$f(t,n) = e^{-i\omega t(e_n - e_{n-1})}\sqrt{\frac{e_n}{n}}, \qquad t \text{ being fixed,} \tag{224}$$

and

$$\widetilde{f}(t,n) = e^{-i\omega t(\varepsilon_n - \varepsilon_{n-1})}\sqrt{\frac{\varepsilon_n}{n}}, \qquad t \text{ being fixed,} \tag{225}$$

respectively. Setting $t = 0$ in the above two equations, we will obtain the nonlinearity functions corresponding to the states $|z\rangle$ in (201) and $|\widetilde{z}\rangle$ in (211), respectively. Now the necessary tools for reproducing all the states (201), (211), (204) and (212) by the displacement operator definition illustrated in section is being in hand.

7. Conclusions

In this tutorial I devised two different approaches deal with unifying the various types of generalized CSs (have been called the mathematical physics CSs) based on different analytical methods recently developed. In the introduction of these states the known three-fold generalizations has never been used. So, we outline a natural questions and try to answer it: is there any probable link between the mathematical physics generalizations of CSs in recent decade and the three-fold generalizations? In this relation, I found the lost ring which connects the above two general categories by two methods. a) The first is based on the conjecture that all of the mathematical physics generalizations in recent decade may be classified in the nonlinear CSs category. The nonlinear CSs method provides a suitable and simple way to obtain the (deformed) annihilation, creation, displacement and Hamiltonian operators, after obtaining the nonlinearity function associated with each set of the mathematical physics generalized CSs. In this manner, the connection with three-fold generalizations has been very clear. b) By the second approach which deals with the related Hilbert space of each set of generalized CSs, it is demonstrated that all the so-called nonlinear CSs (which by the first approach have been established that it contains all the mathematical physics generalized CSs introduced in recent decade), as well as a large class of the previously known CSs in the physical literature such as the photon-added CSs and the binomial states will be obtained by change of bases in the underlying Hilbert spaces. Apart from the above results, using the above two approaches I found a vast new classes of generalized CSs, known as the dual family related to each of them. Due to the fact that the two formalism can not produce GKCSs and the related dual family, to overcome this problem I present a rather different method. So, besides the above mentioned results, I introduced an operator, $\hat{S}(\alpha)$, which its action on any nonlinear CS, transfers it to a situation that enjoys the temporal stability property. We use this operator to obtain the GKCSs, as well as the dual family of them in a consistent way. Also I found a way to introduce the temporally stable or Gazeau-Klauder type of nonlinear CSs. Finally I applied this procedure to some quantum systems with known discrete spectrum, as some physical appearance and introduce the dual family GKCSs associated with them.

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